



On pointwise error estimates for Voronoï-based finite volume methods for the Poisson equation on the sphere

Leonardo A. Poveda¹ · Pedro Peixoto²

Received: 21 June 2022 / Accepted: 13 April 2023 / Published online: 7 June 2023

© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2023

Abstract

In this paper, we give pointwise estimates of a Voronoï-based finite volume approximation of the Laplace-Beltrami operator on Voronoï-Delaunay decompositions of the sphere. These estimates are the basis for local error analysis, in the maximum norm, of the approximate solution of the Poisson equation and its gradient. Here, we consider the Voronoï-based finite volume method as a perturbation of the finite element method. Finally, using regularized Green's functions, we derive quasi-optimal convergence order in the maximum-norm with minimal regularity requirements. Numerical examples show that the convergence is at least as good as predicted.

Keywords Laplace-Beltrami operator · Poisson equation · Spherical icosahedral grids · Finite volume method · *a priori* error estimates · Pointwise estimates · Uniform error estimates

Mathematics Subject Classification (2010) 58J32 · 65N08 · 65N15

1 Introduction

The study of numerical solutions of Partial Differential Equations (PDEs) posed on spheres and other surfaces arise naturally in many applications. For example, it is essential in geophysical fluid dynamics, where a sphere is usually adopted as a domain. For instance, in weather forecasting and climate modeling, PDEs are used and discretized through finite differences, finite elements, and finite volume methods on a spherical domain [27, 30, 55].

Communicated by: Aihui Zhou

✉ Leonardo A. Poveda
lpoveda@math.cuhk.edu.hk

¹ Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong SAR

² Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil

The efficiency and accuracy of the approximate solutions depend on certain characteristics of the discretization of the sphere and the differential operators. In the present work, we start by looking at the approximations of solutions of the Poisson problem on the unit sphere using *spherical icosahedral geodesic grids* [4, 30, 51, 56], i.e., grids that come from an icosahedron inscribed on the sphere, with its vertices and faces projected onto the spherical surface. We obtain a spherical triangular grid whose edges are geodesic arcs and satisfy the so-called Delaunay criterion i.e., maximizing the smallest angle [26, 28, 32]. Following [2, 50], the mentioned construction will allow us to define two types of grids on \mathbb{S}^2 : a triangular Delaunay (primal) decomposition and a Voronoi (dual) decomposition. In what follows, we will consider such grids more generally, not necessarily restricted to those directly built from an icosahedron. Therefore, we will refer to them as general Voronoi-Delaunay decompositions of the sphere throughout this paper.

Voronoi-based finite volume methods are very popular and allow great flexibility. However, the finite volume scheme may not be formally consistent, yet can still lead to second-order convergence results, leading to what is sometimes called supra-convergence [3, 6, 14, 15, 37, 44, 48]. Error estimates of the first order of convergence for approximate solutions of planar Voronoi-based finite volume method in the H^1 and L^2 -norm have been reported by [19, 23–25, 45]. Second-order accuracy in the L^2 -norm using planar dual Donald decompositions, which uses the triangle barycenters as vertices of the dual grid, were considered in [10, 11, 33, 42] and for general surfaces in [34, 35]. These latter works explicitly use properties of the barycentric dual cells, i.e., the quadratic order is obtained by using the centroid of triangles, and therefore these constructions cannot be generally extended to the arbitrary Voronoi-Delaunay decompositions.

Previous work [8, 10, 42, 57] also usually have an additional requirement in the regularity of the exact solution (belong to H^3 or $W^{3,p}$), which is excessive compared to the requirements in finite element methods [7, 12, 49]. However, [22] has reported a sufficient condition to decrease the regularity of the exact solution (is in H^2) but imposes an added regularity requirement on the forcing source term, which is in H^1 to get the optimal convergence order. The authors also highlighted that except for one-dimensional domains or the solution domain has a boundary smooth enough, the H^1 -regularity of the source term does not automatically imply the H^3 -regularity of the exact solution.

On the sphere, [20] show a quadratic order estimate in the Voronoi-Delaunay decomposition on \mathbb{S}^2 in the L^2 -norm and a Spherical Centroidal Voronoi Tessellation (SCVT) optimized grid [17, 18] is used as their Voronoi-Delaunay decomposition and the excessive regularity assumption in the exact solution. Based on this, as a first minor result of this work, we show more general error estimates than given in [20] applying the approach of [22], i.e., decreasing the regularity in the exact solution and imposing a minimum regularity in the source term. Thus we obtain the desired order of convergence. However, we conclude, as in [20], that the proof cannot be extended to Voronoi-Delaunay decompositions in general due to the explicit use of the criteria of the SCVT. In general, determining a quadratic convergence order in the L^2 -norm is an open issue for the Voronoi-based finite volume method. Several efforts have been

made to answer this question and are limited to topological aspects; for example, [47] gives a partial answer and an essential improvement about what is known so far.

The topic of this paper is the convergence analysis for approximate solutions of the Voronoi-based finite volume method in the maximum-norm, extending existing results for the plane [9, 22, 58] to the sphere. We do not know that advances in this direction have been previously described in the literature, particularly for Voronoi-Delaunay decompositions on \mathbb{S}^2 . Therefore, we opted to consider the Voronoi-based finite volume method as a perturbation of the finite element method, an approach widely described in the literature [22, 42, 43].

The main idea is to use the standard error estimation procedures developed for the finite elements on surfaces such as [13, 36, 38] along with the use of the regularized Green's functions on the sphere.

The main result of our work is the proof of the sub-linear convergence order of a classic finite volume method on general spherical Voronoi-Delaunay decompositions, for the Poisson equation, in the maximum norm. This result tightens the gap between theoretical convergence analysis and existing empirical evidence for the convergence of such schemes in the sphere. Empirical evidence indicated the possibility of linear convergence; however, here, our results contain a logarithmic factor caused by the use of linear functions in the primal Delaunay decomposition, which apparently cannot be avoided, as also was initially examined in Euclidean domains via finite element methods by [49, 53, 54] and more recently described in [39–41]. Additionally, linear convergence order in the maximum norm is proved for SCVT grids.

The outline of the paper is as follows: in Section 2, we briefly introduce some notation and the model equation used in this work. Section 3 is devoted to the usual recursive construction of the spherical icosahedral geodesic grids. In Section 4, we establish the classical finite volume method and discrete function spaces. The error estimates for the discrete finite volume scheme are given in Section 5. Section 6 provides numerical experiments and final comments.

2 Problem setting

In this section, we start defining the model problem, some notations, and function spaces that will be used throughout the paper. Let $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \|x\| = 1\}$ be the unit sphere, where $\|\cdot\|$ represents the Euclidean norm in \mathbb{R}^3 . Let ∇_s denote the tangential gradient [21] on \mathbb{S}^2 defined by,

$$\nabla_s u(x) = \nabla u(x) - (\nabla u(x) \cdot \vec{n}_{\mathbb{S}^2, x}) \vec{n}_{\mathbb{S}^2, x},$$

where ∇ denotes the usual gradient in \mathbb{R}^3 and $\vec{n}_{\mathbb{S}^2, x}$ represents the unit outer normal vector to \mathbb{S}^2 at $x = (x_1, x_2, x_3)$. We shall adopt standard notation for Sobolev spaces on \mathbb{S}^2 (see e.g., [29]). Given $1 \leq p \leq \infty$ and k non-negative integer, we denote the Sobolev spaces by,

$$L^p(\mathbb{S}^2) = \left\{ u(x) : \int_{\mathbb{S}^2} |u(x)|^p ds(x) < \infty \right\},$$

$$W^{k,p}(\mathbb{S}^2) = \left\{ u \in L^p(\mathbb{S}^2) : \nabla_s^\alpha u \in L^p(\mathbb{S}^2), \text{ for } 0 \leq |\alpha| \leq k \right\},$$

where $\nabla_s^\alpha = \nabla_{s,1}^{\alpha_1} \nabla_{s,2}^{\alpha_2} \nabla_{s,3}^{\alpha_3}$ is the multi-index notation for the weak tangential derivatives up to order k , where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a vector of non-negative integers with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. The function space $W^{k,p}(\mathbb{S}^2)$ is equipped with the norm

$$\|u\|_{W^{k,p}(\mathbb{S}^2)} = \begin{cases} \left(\sum_{0 \leq |\alpha| \leq k} \|\nabla_s^\alpha u\|_{L^p(\mathbb{S}^2)}^p \right)^{1/p}, & \text{for } 1 \leq p < \infty \\ \max_{0 \leq |\alpha| \leq k} \|\nabla_s^\alpha u\|_{L^\infty(\mathbb{S}^2)}, & \text{for } p = \infty. \end{cases}$$

We set $H^k(\mathbb{S}^2) = W^{k,2}(\mathbb{S}^2)$ along with the standard inner product

$$(u, v) = \int_{\mathbb{S}^2} u(x)v(x)ds(x), \quad \text{for all } u, v \in L^2(\mathbb{S}^2),$$

where $ds(x)$ is the surface area measure. Additionally, we define the zero-averaged subspace of $H^1(\mathbb{S}^2)$ as

$$H_0^1(\mathbb{S}^2) := \left\{ u \in H^1(\mathbb{S}^2) : \int_{\mathbb{S}^2} u(x)ds(x) = 0 \right\},$$

equipped with the H^1 -norm. Throughout this paper, we use C_* as a generic positive real constant, which may vary with the context and depends on the problem data and other model parameters. Also, we will use the notation $a \preceq b$, which means that there is a positive constant C , independent of the mesh size, such that $a \leq Cb$.

We now introduce the Poisson equation to be considered. Let $f \in L^2(\mathbb{S}^2)$ be a given forcing (source) satisfying the compatibility condition,

$$\int_{\mathbb{S}^2} f(x)ds(x) = 0. \quad (2.1)$$

The model problem consists of finding a scalar function $u : \mathbb{S}^2 \rightarrow \mathbb{R}$ satisfying

$$-\Delta_s u(x) = f(x), \quad \text{for each } x \in \mathbb{S}^2, \quad (2.2)$$

where $-\Delta_s = -\nabla_s \cdot \nabla_s$ denotes the Laplacian on \mathbb{S}^2 . We impose $\int_{\mathbb{S}^2} u(x)ds(x) = 0$ to ensure uniqueness of solution. For any $u, v \in H_0^1(\mathbb{S}^2)$, we define the bilinear functional $\mathcal{A} : H_0^1(\mathbb{S}^2) \times H_0^1(\mathbb{S}^2) \rightarrow \mathbb{R}$ such that

$$\mathcal{A}(u, v) = \int_{\mathbb{S}^2} \nabla_s u(x) \cdot \nabla_s v(x)ds(x). \quad (2.3)$$

The bilinear functional is well-defined on the space $H_0^1(\mathbb{S}^2) \times H_0^1(\mathbb{S}^2)$ and is continuous and coercive, i.e., there are positive constants C_0 and C_1 such that

$$\begin{aligned} |\mathcal{A}(u, v)| &\leq C_0 \|\nabla_s u\|_{L^2(\mathbb{S}^2)} \|\nabla_s v\|_{L^2(\mathbb{S}^2)}, \quad \text{for each } u, v \in H_0^1(\mathbb{S}^2), \\ \mathcal{A}(u, u) &\geq C_1 \|\nabla_s u\|_{L^2(\mathbb{S}^2)}^2, \quad \text{for each } u \in H_0^1(\mathbb{S}^2). \end{aligned}$$

The variational formulation of (2.2) reads: find $u \in H_0^1(\mathbb{S}^2)$ such that

$$\mathcal{A}(u, v) = (f, v), \quad \text{for each } v \in H_0^1(\mathbb{S}^2), \quad (2.4)$$

in which $(f, v) = \int_{\mathbb{S}^2} f(x)v(x)ds(x)$ and the source data f satisfies (2.1).

As a consequence of the Lax-Milgram theorem [7, 12], problem (2.4) has a unique solution. This solution is such that for some positive constant C_S ,

$$\|\nabla_s u\|_{L^2(\mathbb{S}^2)} \leq C_S \|f\|_{L^2(\mathbb{S}^2)}. \quad (2.5)$$

Moreover, we have the regularity property: for $f \in L^2(\mathbb{S}^2)$ satisfying (2.1), the unique weak solution $u \in H^2(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$ of (2.4) satisfies, for some positive constant C_R ,

$$\|u\|_{H^2(\mathbb{S}^2)} \leq C_R \|f\|_{L^2(\mathbb{S}^2)}. \quad (2.6)$$

A detailed proof of (2.6) can be found in [21, Theorem 3.3 pp. 304].

3 Spherical icosahedral geodesic grids

In this section, we describe the discretization framework to approximate the sphere \mathbb{S}^2 , following [4, 27, 30]. The spherical icosahedral grid can be constructed by defining an icosahedron inscribed inside \mathbb{S}^2 , which has triangular faces and vertices. Each edge of the original icosahedron whose vertices are on \mathbb{S}^2 is projected onto the surface of \mathbb{S}^2 . Then, we employ a recursive refinement of the grid by connecting the midpoints of the geodesic arcs to generate four sub-triangles in each geodesic triangle. This procedure may be applied to all geodesic triangles of the initial icosahedron to create a grid of desired resolution (see Fig. 1).

3.1 Voronoï-Delaunay decomposition

Let $d(x, y)$ denote the geodesic distance between x and y on \mathbb{S}^2 , defined by

$$d(x, y) := \arccos \langle x, y \rangle_{\mathbb{R}^3} \in [0, \pi],$$

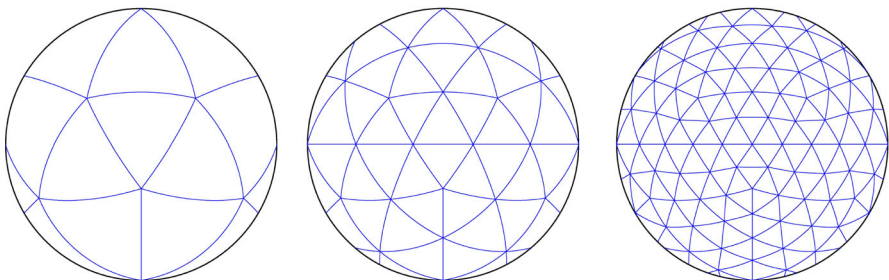


Fig. 1 Spherical icosahedral grids with levels 0, 1 and 2 with 12, 42 and 162 vertices, respectively

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ denotes the Euclidean scalar product in \mathbb{R}^3 . We will use the notation $m_a(\cdot)$ and $m_l(\cdot)$ for the standard measures of superficial area and curve length, respectively. Let $S_N = \{x_i\}_{i=1}^N$ denote the set of distinct vertices on \mathbb{S}^2 , where $N = 2^{2\ell} \cdot 10 + 2$ is the number of vertices and ℓ is the level of grid refinement [4]. We denote \tilde{T}_{ijk} as a geodesic triangle with vertices $x_i, x_j, x_k \in S_N$ and define the spherical Delaunay (primal) decomposition on \mathbb{S}^2 as the set $\tilde{T}_h := \{\tilde{T}_{ijk} : ijk \in \Sigma\}$, where Σ is the set of indices such that i, j, k are adjacent neighbors in S_N . Here the subscript h denotes the main grid parameter to be defined later.

The dual Voronoï decomposition of \tilde{T}_h is constructed following [2, 50]. For each vertex $x_i \in S_N$, $1 \leq i \leq N$, its associated Voronoï cell \tilde{V}_i is given by

$$\tilde{V}_i := \left\{ x \in \mathbb{S}^2 : d(x, x_i) < d(x, x_j), \text{ for each } 1 \leq j \leq N, \text{ and } j \neq i \right\}.$$

Each Voronoï cell \tilde{V}_i consists of all points $x \in \mathbb{S}^2$ closer to x_i than any other vertex of S_N . Voronoï cells are open and convex polygons on \mathbb{S}^2 , limited by geodesic arcs at their boundaries. In particular, every two cells have an empty intersection, and $\bigcup_{i=1}^N \text{cl}(\tilde{V}_i) = \mathbb{S}^2$, where $\text{cl}(\cdot)$ denotes the closure of the cell. Further, given two adjacent vertices x_i and x_j , we denote by $\tilde{\Gamma}_{ij} = \text{cl}(\tilde{V}_i) \cap \text{cl}(\tilde{V}_j) \neq \emptyset$ the geodesic Voronoï edge on \mathbb{S}^2 associated to the vertices x_i and x_j . Thus, for each vertex x_i we can denote the set of indices of its neighbors x_j such that $m_l(\tilde{\Gamma}_{ij}) > 0$, i.e.,

$$\Lambda(i) = \{j : j \neq i \text{ and } \tilde{\Gamma}_{ij} = \text{cl}(\tilde{V}_i) \cap \text{cl}(\tilde{V}_j) \neq \emptyset\}.$$

Each \tilde{V}_i has smooth piecewise boundary $\partial\tilde{V}_i$ formed by the Voronoï dual edges $\tilde{\Gamma}_{ij}$, with $j \in \Lambda(i)$, i.e., $\partial\tilde{V}_i = \bigcup_{j \in \Lambda(i)} \tilde{\Gamma}_{ij}$. For x_i and x_j neighboring vertices with $j \in \Lambda(i)$, we denote by $\tilde{\tau}_{ij}$ the Delaunay edge joining vertices x_i and x_j , and by x_{ij} and m_{ij} the midpoints of the geodesic edges $\tilde{\tau}_{ij}$ (Delaunay) and $\tilde{\Gamma}_{ij}$ (Voronoi) respectively. By construction, each geodesic Delaunay edge $\tilde{\tau}_{ij}$ is perpendicular to geodesic Voronoï edge $\tilde{\Gamma}_{ij}$ and the plane formed by $\tilde{\Gamma}_{ij}$ and the origin, bisects $\tilde{\tau}_{ij}$ at its midpoint x_{ij} (see [19, 50]). Therefore $d(x_i, x) = d(x, x_j)$, for each $x \in \tilde{\Gamma}_{ij}$, and we denote by $\tilde{n}_{x, \tilde{\Gamma}_{ij}}$ the co-normal unit vector at the Voronoï edge $\tilde{\Gamma}_{ij}$ lying in the plane $\mathbb{T}_{x, \mathbb{S}^2}$ tangent to \mathbb{S}^2 at x . Finally, $\tilde{n}_{x, \tilde{\Gamma}_{ij}}$ is parallel to $\overrightarrow{x_i x_j}$, i.e., $\tilde{n}_{x, \tilde{\Gamma}_{ij}} \parallel \overrightarrow{x_i x_j}$ for each $x \in \tilde{\Gamma}_{ij}$, see Fig. 2.

Remark 3.1 The Voronoï-Delaunay decomposition constructed as described above will be called a non-optimized grid and denoted throughout this paper as Voronoï-Delaunay decomposition NOPT.

Remark 3.2 In the NOPT decomposition, the constrained cell centroids normally do not coincide with the nodes. However, this can be an important property for discretization. Therefore, we will also consider Spherical Centroidal Voronoï Tessellations (SCVT). These are constructed through an iterative method (Lloyd's algorithm, see [18, 19]). In the SCVT tessellation, the vertex generating each cell $\tilde{V}_i \subset \mathbb{S}^2$, is the

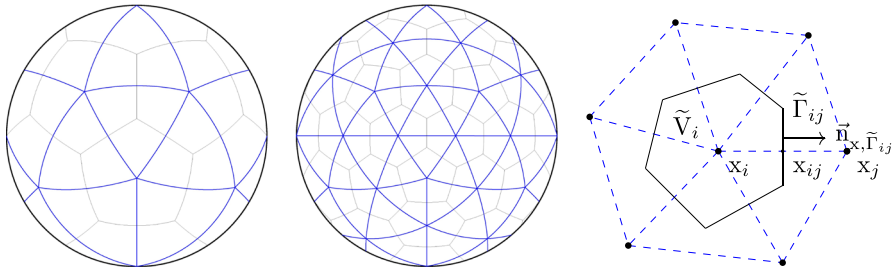


Fig. 2 Voronoï-Delaunay decomposition on \mathbb{S}^2 with levels 0 and 1, and the geometric configuration of Voronoï cell and its associated triangles

constrained cell centroid x_i^c , i.e., the minimum of the following function

$$F(x) = \int_{\tilde{V}_i} \|y - x\|^2 ds(y).$$

Given a Voronoï-Delaunay decomposition, we consider some grid parameters previously defined in [19, 20]. Let $h_i = \max_{x \in \tilde{V}_i} d(x_i, x)$ be the maximum geodesic distance between vertex x_i and the points in its associated cell \tilde{V}_i and $h = \max_{i=1, \dots, N} h_i$.

In addition, we consider the following shape regularity or almost uniform conditions given by [12, 42, 45]:

Definition 3.3 (Almost uniform) We say that a Voronoï-Delaunay decomposition $\tilde{\mathcal{V}}_h = \{x_i, \tilde{V}_i\}_{i=1}^N$ on \mathbb{S}^2 is finite volume regular if for every spherical polygon $\tilde{V}_i \in \tilde{\mathcal{V}}_h$ with boundary $\partial \tilde{V}_i = \bigcup_{j \in \Lambda(i)} \tilde{\Gamma}_{ij}$, there exist positive constants C_0 and C_1 , independent of h such that

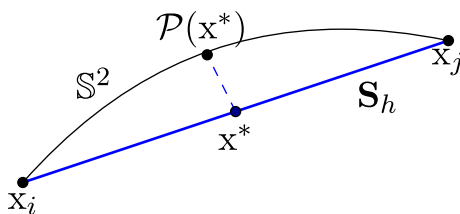
$$\frac{1}{C_0} h \leq m_l(\tilde{\Gamma}_{ij}) \leq C_0 h, \quad \text{and} \quad \frac{1}{C_1} h^2 \leq m_a(\tilde{V}_i) \leq C_1 h^2.$$

From now onward, we assume that the Voronoï-Delaunay decompositions NOPT and SCVT are almost uniform grids on \mathbb{S}^2 .

3.2 Geometric correspondence

We describe here geometric relations between \mathbb{S}^2 and its polyhedral approximation \mathbf{S}_h , using the framework given by [13, 34]. First, we assume that \mathbb{S}^2 is approximated by a polyhedral sequence \mathbf{S}_h formed by the decomposition \mathcal{T}_h (planar triangles) as h goes to zero. The smooth and bijective mapping $\mathcal{P} : \mathbf{S}_h \rightarrow \mathbb{S}^2$ is defined as radial projection of any point $x^* \in \mathbf{S}_h$ onto the spherical surface, i.e., $\mathcal{P}(x^*) = x^* / \|x^*\|$ (Fig. 3). Observe that, by construction, the vertices ($S_N = \{x_i\}_{i=1}^N$) of the polyhedral \mathbf{S}_h belong to the surface of \mathbb{S}^2 i.e., $S_N = \mathbf{S}_h \cap \mathbb{S}^2$. This implies that $\mathbf{S}_h = \bigcup_{ijk \in \Sigma} T_{ijk}$ and $\mathbb{S}^2 = \bigcup_{ijk \in \Sigma} \tilde{T}_{ijk}$, where Σ is the set of neighboring vertices in S_N . Notice that the polyhedral of level zero is the initial icosahedron.

Fig. 3 Radial projection of polyhedral \mathbf{S}_h on sphere \mathbb{S}^2



We also consider the following spherical shell in \mathbb{R}^3 :

$$\Omega_h := \left\{ x^* \in \mathbb{R}^3 \setminus \{0\} : 1 - Ch^2 < \|x^*\| < 1 + Ch^2 \right\},$$

with C chosen such that \mathbb{S}^2 and \mathbf{S}_h are contained in Ω_h . For functions $u \in H^2(\mathbb{S}^2)$, we denote by u^Ω the extension of u to Ω_h given by $u^\Omega(x^*) = u(\mathcal{P}(x^*))$, for each $x^* \in \mathbf{S}_h$. The following result has been shown in [19, Proposition 1, pp. 1677]:

Proposition 3.4 *For any $x^* \in \Omega_h$ and $x = \mathcal{P}(x^*) \in \mathbb{S}^2$, and $i, j \in \{1, 2, 3\}$,*

$$\begin{aligned} \nabla_s u(x) &= \nabla u^\Omega(x), \quad \nabla(\partial_i u^\Omega(x)) = \nabla_s(\partial_{s,i} u(x)) - (\partial_{s,i} u(x)) \vec{n}_{\mathbb{S}^2, x}, \\ \|x^*\| \nabla u^\Omega(x^*) &= \nabla u^\Omega(x), \quad \|x^*\|^2 \partial_i \partial_j u^\Omega(x^*) = \partial_i \partial_j u^\Omega(x), \end{aligned}$$

where ∂_i denotes partial derivative with respect to x_i and $\partial_{s,i}$ denotes the i -th component of the tangential derivative, for $i = 1, 2, 3$.

The following result compares the norms of functions defined on \mathbb{S}^2 and \mathbf{S}_h . The proof is given in [13].

Proposition 3.5 *Let $u \in W^{2,p}(\mathbb{S}^2)$ with $1 \leq p \leq \infty$. There exist positive constants C_0, C_1 and C_2 such that for h small enough,*

$$\begin{aligned} \frac{1}{C_0} \|u\|_{L^p(\tilde{\Gamma}_{ijk})} &\leq \|\bar{u}^\Omega\|_{L^p(\Gamma_{ijk})} \leq C_0 \|u\|_{L^p(\tilde{\Gamma}_{ijk})}, \\ \frac{1}{C_1} \|\nabla_s u\|_{L^p(\tilde{\Gamma}_{ijk})} &\leq \|\nabla \bar{u}^\Omega\|_{L^p(\Gamma_{ijk})} \leq C_1 \|\nabla_s u\|_{L^p(\tilde{\Gamma}_{ijk})}, \\ \|\nabla^\alpha \bar{u}^\Omega\|_{L^p(\Gamma_{ijk})} &\leq C_2 \sum_{0 \leq |\alpha| \leq 2} \|\nabla_s^\alpha u\|_{L^p(\tilde{\Gamma}_{ijk})}, \end{aligned}$$

where \bar{u}^Ω is the extension of u to Ω_h restricted to \mathbf{S}_h , and $\nabla^\alpha, \nabla_s^\alpha$ denote the usual derivatives and tangential derivatives up to order 2.

4 A Voronoï-based finite volume method

In this section, we seek an approximate solution of (2.2) via a finite difference/volume scheme. First, from Gauss theorem, we have that

$$-\int_{\tilde{V}_i} \Delta_s u(x) ds(x) = -\int_{\partial \tilde{V}_i} \nabla_s u(x) \cdot \vec{n}_{x, \tilde{V}_i} d\gamma(x),$$

where \vec{n}_{x, \tilde{V}_i} denotes the co-normal unit vector on $\partial \tilde{V}_i$ at x , and $d\gamma(x)$ is the geodesic length measure. Since $\partial \tilde{V}_i = \cup_{j \in \Lambda(i)} \tilde{\Gamma}_{ij}$, we have that

$$-\sum_{j \in \Lambda(i)} \int_{\tilde{\Gamma}_{ij}} \nabla_s u(x) \cdot \vec{n}_{x, \tilde{\Gamma}_{ij}} d\gamma(x) = \int_{\tilde{V}_i} f(x) ds(x).$$

We denote the continuous flux of u across the edge $\tilde{\Gamma}_{ij}$ by

$$\tilde{\mathcal{F}}_{ij}(u) := -\int_{\tilde{\Gamma}_{ij}} \nabla_s u(x) \cdot \vec{n}_{x, \tilde{\Gamma}_{ij}} d\gamma(x), \quad (4.1)$$

and define its central difference approximation

$$\overline{\mathcal{F}}_{ij}(u_h) := -m_l(\tilde{\Gamma}_{ij}) \frac{u_h(x_j) - u_h(x_i)}{\|x_j - x_i\|}. \quad (4.2)$$

Additionally, for each cell \tilde{V}_i , let f_i denote the mean value of the data f on \tilde{V}_i , i.e.,

$$f_i = \frac{1}{m_a(\tilde{V}_i)} \int_{\tilde{V}_i} f(x) ds(x). \quad (4.3)$$

The Voronoï-based finite volume scheme is defined as

$$(\mathbf{L}_{s,h}(u_h))_i := \frac{1}{m_a(\tilde{V}_i)} \sum_{j \in \Lambda(i)} \overline{\mathcal{F}}_{ij}(u_h) = f_i, \quad \text{for } 1 \leq i \leq N, \quad (4.4)$$

where $\mathbf{L}_{s,h}$ is a discretization of the Laplacian, u_h is an approximate solution and f_i is defined in (4.3). To emphasize the dependence of the grid, we always will use the subscript h . The finite volume scheme (4.4) is conservative, since we have that,

$$\sum_{i=1}^N \sum_{j \in \Lambda(i)} \overline{\mathcal{F}}_{ij}(u_h) = 0,$$

which follows from $\overline{\mathcal{F}}_{ij} = -\overline{\mathcal{F}}_{ji}$ if the vertices x_i and x_j are neighbors with $m_l(\tilde{\Gamma}_{ij}) > 0$.

4.1 Discrete function spaces

In this subsection, we introduce some local function spaces on \mathbb{S}^2 , as in [13, 21, 34]. We define the Lagrange finite element space on a polyhedral \mathbf{S}_h as

$$\mathbb{P}_1(\mathbf{S}_h, \mathcal{T}_h) := \left\{ \bar{u}_h \in C^0(\mathbf{S}_h) : \bar{u}_h|_{T_{ijk}}, \text{ is linear affine for each } T_{ijk} \in \mathcal{T}_h \right\},$$

and the corresponding lifted finite element space on \mathbb{S}^2 ,

$$\mathbb{P}_1(\mathbb{S}^2, \tilde{\mathcal{T}}_h) := \left\{ u_h \in C^0(\mathbb{S}^2) : u_h = \bar{u}_h \circ \mathcal{P}^{-1}, \text{ for each } \bar{u}_h \in \mathbb{P}_1(\mathbf{S}_h, \mathcal{T}_h) \right\},$$

where \mathcal{P}^{-1} denotes the inverse of the radial projection \mathcal{P} . For the approximation of functions on $H_0^1(\mathbb{S}^2)$, we consider \tilde{U}_h the zero-averaged subspace of $\mathbb{P}_1(\mathbb{S}^2, \tilde{\mathcal{T}}_h)$ given by

$$\tilde{U}_h(\mathbb{S}^2) := \left\{ u_h \in \mathbb{P}_1(\mathbb{S}^2, \tilde{\mathcal{T}}_h) : \int_{\mathbb{S}^2} u_h(x) ds(x) = 0 \right\}.$$

Notice that, from Proposition 3.4, we can get $\tilde{U}_h \subset H_0^1(\mathbb{S}^2)$ endowed with the H^1 -norm. As in [42, 43], we define the piecewise constant function space associated with the dual Voronoi decomposition given by

$$\tilde{V}_h(\mathbb{S}^2) := \left\{ v \in L^2(\mathbb{S}^2) : v|_{\tilde{V}_i}, \text{ is constant in each } \tilde{V}_i \text{ for } 1 \leq i \leq N \right\}.$$

We introduce interpolation operators $\tilde{\Pi}_h$ and $\tilde{\Pi}_h^*$ mapping functions defined on \mathbb{S}^2 onto \tilde{U}_h and \tilde{V}_h , respectively. Note that, given the function values at the vertices of the Voronoi grid, the operators are uniquely defined. The following interpolation estimates will be used in our analysis and are shown in [20, Proposition 3, pp. 1677].

Proposition 4.1 (Interpolation estimates) *Assume $u \in W^{2,p}(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$ for $2 \leq p \leq \infty$ and $v \in H^2(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$. Then, for h small enough, there exist positive constants C_U and C_V independent of h such that,*

$$\begin{aligned} \|u - \tilde{\Pi}_h(u)\|_{W^{k,p}(\mathbb{S}^2)} &\leq C_U h^{2-k} \|u\|_{W^{2,p}(\mathbb{S}^2)}, \quad \text{for each } k \in \{0, 1\}, \\ \|v - \tilde{\Pi}_h^*(v)\|_{L^2(\mathbb{S}^2)} &\leq C_V h \|v\|_{H^2(\mathbb{S}^2)}. \end{aligned}$$

We also define the linear transference interpolation $\tilde{\mathcal{I}}_h : \tilde{U}_h \rightarrow \tilde{V}_h$ as

$$\tilde{\mathcal{I}}_h(u_h)(x) = \sum_{i=1}^N u_h(x_i) \chi_i(x), \quad \text{for each } u_h \in \tilde{U}_h,$$

where χ_i represents the characteristic function corresponding to the cell \tilde{V}_i , with $1 \leq i \leq N$.

In order to show basic estimates, we shall need the following inverse estimate for finite element functions (see e.g., [7, 12, 13]), which follows from the almost-uniformity of the decomposition $\tilde{\mathcal{T}}_h$. The proof is valid on \mathbb{S}^2 under conditions of Propositions 3.4 and 3.5, cf. [13, Proposition 2.7, pp. 812] or [36, Lemma 3.4, pp. 524].

Lemma 4.2 (Inverse estimate) *Let $\tilde{\mathcal{T}}_h$ be an almost uniform Voronoi-Delaunay decomposition on \mathbb{S}^2 . Assume that l, m are non negative integers with $l \leq m$ and $1 \leq p, q \leq \infty$, such that $\tilde{U}_h \subset W^{l,p}(\tilde{\mathcal{T}}_{ijk}) \cap W^{m,q}(\tilde{\mathcal{T}}_{ijk})$. Then, there exists a positive constant C independent of h , such that $v_h \in \tilde{U}_h$ satisfies,*

$$\|v_h\|_{W^{m,p}(\tilde{\mathcal{T}}_{ijk})} \leq Ch^{l-m-2(1/q-1/p)} \|v_h\|_{W^{l,q}(\tilde{\mathcal{T}}_{ijk})}.$$

We establish the following auxiliary result for later use in the analysis.

Lemma 4.3 *Let $\tilde{\mathcal{T}}_{ijk} \in \tilde{\mathcal{T}}_h$ be a geodesic triangle of an almost uniform Voronoi-Delaunay decomposition on \mathbb{S}^2 and $\tilde{\tau}_{ij} \subset \partial\tilde{\mathcal{T}}_{ijk}$. Then, for $1 \leq q < \infty$ and $v_h \in \tilde{U}_h$, there exists a positive constant C independent of h , such that*

$$\int_{\tilde{\tau}_{ij}} [v_h(x) - \tilde{I}_h(v_h)(x)] d\gamma(x) = 0, \quad (4.5a)$$

$$\|v_h - \tilde{I}_h(v_h)\|_{L^q(\tilde{\mathcal{T}}_{ijk})} \leq Ch \|v_h\|_{W^{1,q}(\tilde{\mathcal{T}}_{ijk})}. \quad (4.5b)$$

Proof Let x_{ij} be the midpoint of $\tilde{\tau}_{ij} \subset \partial\tilde{\mathcal{T}}_{ijk}$, we define $\tilde{\tau}_{ij}^{(n)}$ as the geodesic between the points x_{ij} and x_n with $n \in \{i, j\}$. We know that $\tilde{I}_h(v_h)(x) = v_h(x_n)$ for each $x \in \tilde{\tau}_{ij}^{(n)}$, where $n \in \{i, j\}$ and $\tilde{\tau}_{ij} = \tilde{\tau}_{ij}^{(i)} \cup \tilde{\tau}_{ij}^{(j)}$. We can immediately derive the following estimates,

$$\begin{aligned} \int_{\tilde{\tau}_{ij}} [v_h(x) - \tilde{I}_h(v_h)(x)] d\gamma(x) &= \int_{\tilde{\tau}_{ij}^{(i)}} [v_h(x) - v_h(x_i)] d\gamma(x) + \int_{\tilde{\tau}_{ij}^{(j)}} [v_h(x) - v_h(x_j)] d\gamma(x) \\ &= \int_{\tilde{\tau}_{ij}} v_h(x) d\gamma(x) - \int_{\tilde{\tau}_{ij}^{(i)}} [v_h(x_i) + v_h(x_j)] d\gamma(x) \\ &= \int_{\tilde{\tau}_{ij}} v_h(x) d\gamma(x) - \frac{1}{2} m_l(\tilde{\tau}_{ij}) [v_h(x_i) + v_h(x_j)], \end{aligned} \quad (4.6)$$

where $m_l(\tilde{\tau}_{ij})$ denotes the length of $\tilde{\tau}_{ij}$. Let v_h^Ω be the lift of v_h to the spherical shell $\Omega_{ijk} := \{x^* \in \Omega_h : \mathcal{P}(x^*) \in \tilde{\mathcal{T}}_{ijk}\}$, with $v_h(x) = v_h^\Omega(x)$ for all $x \in \mathbb{S}^2$. Also let x_{ij}^* be the midpoint of $[x_i, x_j]$ (the segment goes from x_i to x_j) such that $x_{ij} = \mathcal{P}(x_{ij}^*)$. Now, assume that $v_h^\Omega \in C^1(\Omega_{ijk})$, by Taylor's Theorem for v_h^Ω around x_{ij} and integrating over $\tilde{\tau}_{ij}$, we obtain

$$\begin{aligned} \int_{\tilde{\tau}_{ij}} v_h(x) d\gamma(x) &= \int_{\tilde{\tau}_{ij}} v_h^\Omega(x) d\gamma(x) = m_l(\tilde{\tau}_{ij}) v_h^\Omega(x_{ij}) \\ &\quad + \int_{\tilde{\tau}_{ij}} \int_0^1 \nabla v_h^\Omega(x_{ij} + t(x - x_{ij})) \cdot (x - x_{ij}) dt d\gamma(x). \end{aligned}$$

The second term in the right-hand side vanishes by parity of $\nabla v_h^\Omega(y)$ for any $y \in [x, x_{ij}]$ and a symmetry of $\tilde{\tau}_{ij}$ with respect to the midpoint x_{ij} . It follows that,

$$\int_{\tilde{\tau}_{ij}} v_h(x) d\gamma(x) = m_l(\tilde{\tau}_{ij}) v_h^\Omega(x_{ij}).$$

Substituting the expression above into (4.6), one obtains

$$\int_{\tilde{\tau}_{ij}} [v_h(x) - \tilde{I}_h(v_h)(x)] d\gamma(x) = m_l(\tilde{\tau}_{ij}) v_h^\Omega(x_{ij}) - \frac{1}{2} m_l(\tilde{\tau}_{ij}) [v_h(x_i) + v_h(x_j)] = 0,$$

which follows from $v_h^\Omega(x_{ij}) = v_h^\Omega(x_{ij}^*)$ and $v_h^\Omega(x_{ij}^*) = \frac{1}{2} [v_h(x_i) + v_h(x_j)]$. This shows the identity (4.5a). For (4.5b), we consider \tilde{Q}_i, \tilde{Q}_j and \tilde{Q}_k the three spherical polygonal regions makeup by the intersection of the triangle \tilde{T}_{ijk} with the Voronoï cells associated to each vertex x_i, x_j and x_k , i.e.,

$$\tilde{Q}_n = \tilde{T}_{ijk} \cap \tilde{V}_n, \quad \text{for } n \in \{i, j, k\}.$$

For $x \in \tilde{Q}_i$ we have $\tilde{I}_h(v_h)(x) = v_h(x_i)$. Then

$$\begin{aligned} |v_h(x) - v_h(x_i)| &= \left| \int_0^1 \nabla v_h^\Omega(x_i + t(x - x_i)) \cdot (x - x_i) dt \right| \\ &\leq \|x - x_i\| \max_{x^* \in [x, x_i]} \|\nabla v_h^\Omega(x^*)\|, \end{aligned}$$

where v_h^Ω is the radial extension of v_h to the spherical shell Ω_h , we used its Taylor expansion and $[x, x_i]$ is a segment that connects x with x_i . For $1 \leq q < \infty$ and, by integrating over \tilde{Q}_i , we get

$$\begin{aligned} \int_{\tilde{Q}_i} |v_h(x) - \tilde{I}_h(v_h)(x)|^q ds(x) &\leq \int_{\tilde{Q}_i} \|x - x_i\|^q \max_{x^* \in [x, x_i]} \|\nabla v_h^\Omega(x^*)\|^q ds(x) \\ &\leq Ch^q \int_{\tilde{Q}_i} \max_{y \in \tilde{Q}_i} \|\nabla v_h^\Omega(y)\|^q ds(x). \end{aligned}$$

Recalling Definition 3.3 and Lemma 4.2 with $m = l = 1$ and $p = \infty$, we can have

$$\begin{aligned} \|v_h - \tilde{I}_h(v_h)\|_{L^q(\tilde{Q}_i)}^q &\leq Ch^q m_a(\tilde{Q}_i) \|\nabla v_h^\Omega\|_{L^\infty(\tilde{Q}_i)}^q \leq Ch^q h^2 \|\nabla v_h^\Omega\|_{L^\infty(\tilde{Q}_i)}^q \\ &\leq Ch^q h^2 h^{-2} \|\nabla v_h^\Omega\|_{L^q(\tilde{Q}_i)}^q \leq Ch^q \|\nabla v_h^\Omega\|_{L^q(\tilde{Q}_i)}^q. \end{aligned}$$

The estimates similarly hold for \tilde{Q}_j and \tilde{Q}_k . So combining these results, we get that.

$$\|v_h - \tilde{I}_h(v_h)\|_{L^q(\tilde{T}_{ijk})} \leq Ch \|\nabla_s v_h\|_{L^q(\tilde{T}_{ijk})},$$

which completes the proof. \square

Let us introduce some discrete norms and seminorms for functions on \tilde{U}_h . Similarly to [5, 16, 19], for $1 \leq p < \infty$, we denote

$$\begin{aligned}\|u_h\|_{0,p,h}^p &= \sum_{i=1}^N m_a(\tilde{V}_i) |u_h(x_i)|^p, \\ |u_h|_{1,p,h}^p &= \sum_{i=1}^N \sum_{j \in \Lambda(i)} \frac{1}{2} m_l(\tilde{\Gamma}_{ij}) d(x_i, x_j) \left| \frac{u_h(x_i) - u_h(x_j)}{\|x_i - x_j\|} \right|^p, \\ \|u_h\|_{1,p,h}^p &= \|u_h\|_{0,p,h}^p + |u_h|_{1,p,h}^p.\end{aligned}$$

In the case $p = 2$, we omit p in our notation and simply write $\|\cdot\|_{0,2,h} = \|\cdot\|_{0,h}$ and $\|\cdot\|_{1,2,h} = \|\cdot\|_{1,h}$ for the norms and $|\cdot|_{1,2,h} = |\cdot|_{1,h}$ for the seminorm. Furthermore, for the case $p = \infty$, we can use the usual notational convention for the max-norm, i.e., $\|\cdot\|_{L^\infty(\mathbb{S}^2)}$, for functions on \tilde{U}_h .

Proposition 4.4 *For $u_h \in \tilde{U}_h$, there exist positive constants C_0 and C_1 , independent of h such that,*

$$\frac{1}{C_0} \|u_h\|_{0,p,h} \leq \|u_h\|_{L^p(\mathbb{S}^2)} \leq C_0 \|u_h\|_{0,p,h}, \quad (4.7a)$$

$$\frac{1}{C_1} |u_h|_{1,p,h} \leq \|\nabla_s u_h\|_{L^p(\mathbb{S}^2)} \leq C_1 |u_h|_{1,p,h}, \quad (4.7b)$$

with $p \in \{1, 2\}$.

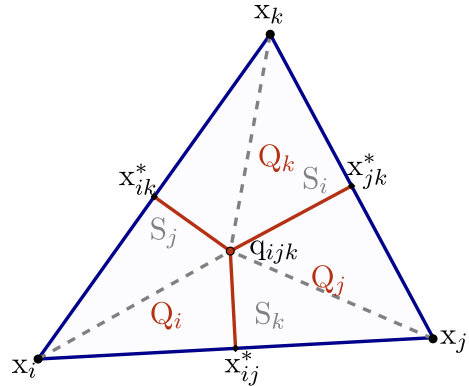
Proof About $p = 2$, we cite [20, Proposition 4, pp. 1678] and [42, Lemma 3.2.1, pp. 124]. About $p = 1$, from Proposition 3.5, for the extension \bar{u}_h^Ω of u_h to Ω_h restricted to S_h , we have

$$\int_{\tilde{T}_{ijk}} |u_h(x)| ds(x) \leq C \int_{T_{ijk}} |\bar{u}_h^\Omega(x^*)| ds(x^*). \quad (4.8)$$

Note that \bar{u}_h^Ω is linear in planar triangle $T_{ijk} \in \mathcal{T}_h$ and assume that $m_a(T_{ijk}) = \sum_{n \in \{i,j,k\}} m_a(S_n)$, where S_n denotes the triangular regions in T_{ijk} , shown enclosed by dashed lines in Fig. 4. Then, by numerical integration formula with second-order accuracy, we compute that

$$\sum_{n \in \{i,j,k\}} \int_{S_n} |\bar{u}_h^\Omega(x^*)| ds(x^*) = m_a(S_k) |\bar{u}_h(x_{ij}^*)| + m_a(S_i) |\bar{u}_h(x_{jk}^*)| + m_a(S_j) |\bar{u}_h(x_{ik}^*)|,$$

Fig. 4 Geometric configuration of planar triangle $T_{ijk} \in \mathcal{T}_h$ with vertices x_i, x_j and x_k and its circumcenter denoted by q_{ijk}



where x_{ij}^*, x_{jk}^* and x_{ik}^* represent the midpoints of each edge of T_{ijk} . Then, we have

$$\begin{aligned} \sum_{n \in \{i, j, k\}} \int_{S_n} |\bar{u}_h^\Omega(x^*)| ds(x^*) &= m_a(S_k) \left| \frac{\bar{u}_h(x_i) + \bar{u}_h(x_j)}{2} \right| + m_a(S_i) \left| \frac{\bar{u}_h(x_j) + \bar{u}_h(x_k)}{2} \right| \\ &\quad + m_a(S_j) \left| \frac{\bar{u}_h(x_i) + \bar{u}_h(x_k)}{2} \right| \\ &\leq \frac{m_a(S_k) + m_a(S_j)}{2} |\bar{u}_h(x_i)| + \frac{m_a(S_k) + m_a(S_i)}{2} |\bar{u}_h(x_j)| \\ &\quad + \frac{m_a(S_i) + m_a(S_j)}{2} |\bar{u}_h(x_k)|. \end{aligned} \quad (4.9)$$

Notice that

$$\frac{m_a(S_k) + m_a(S_j)}{2} = m_a(Q_i), \quad \frac{m_a(S_k) + m_a(S_i)}{2} = m_a(Q_j) \quad \text{and} \quad \frac{m_a(S_i) + m_a(S_j)}{2} = m_a(Q_k).$$

In fact, we have $\tilde{Q}_n = \mathcal{P}(Q_n)$ for $n \in \{i, j, k\}$. Thus, gathering (4.8) and (4.9), and summing up all triangles of $\tilde{\mathcal{T}}_h$, we obtain

$$\|u_h\|_{L^1(\mathbb{S}^2)} = \sum_{\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h} \int_{\tilde{T}_{ijk}} |u_h(x)| ds(x) \leq C \sum_{i=1}^N m_a(\tilde{V}_i) |u_h(x_i)| = C \|u_h\|_{0,1,h},$$

which yields the right-hand side of (4.7a). Similarly, we can obtain the left-hand side. The inequality (4.7b) follows by using the fact that \bar{u}_h is linear on each $T_{ijk} \in \mathcal{T}_h$. Furthermore, $\nabla \bar{u}_h$ is constant on each $T_{ijk} \in \mathcal{T}_h$; then the result is given by using the numerical integration and central difference approximation with the second order of accuracy. \square

4.2 A variational formulation

We now describe a variational formulation for the finite volume scheme. For $(u_h, v_h) \in \tilde{U}_h \times \tilde{U}_h$, we define the total flux bilinear form $\tilde{\mathcal{A}}_h : \tilde{U}_h \times \tilde{U}_h \rightarrow \mathbb{R}$ such that

$$\tilde{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(v_h)) = \sum_{i=1}^N v_h(x_i) \sum_{j \in \Lambda(i)} \tilde{\mathcal{F}}_{ij}(u_h), \quad (4.10)$$

and its discrete version

$$\overline{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(v_h)) = \sum_{i=1}^N v_h(x_i) \sum_{j \in \Lambda(i)} \overline{\mathcal{F}}_{ij}(u_h), \quad (4.11)$$

where $\tilde{\mathcal{F}}_{ij}$ and $\overline{\mathcal{F}}_{ij}$ are defined in (4.1) and (4.2) respectively. So, an approximation $u_h \in \tilde{U}_h$ of (4.4) is defined as the unique solution of the discrete problem: find $u_h \in \tilde{U}_h$ such that

$$\overline{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(v_h)) = (f, \tilde{\mathcal{I}}_h(v_h)), \quad \text{for each } v_h \in \tilde{U}_h, \quad (4.12)$$

In other words, we have, in each Voronoï volume, that $\overline{\mathcal{A}}_h(u_h, \chi_i) = f_i$, where the values of f_i are defined in (4.3).

Proposition 4.5 [19, 45] *Let $\tilde{\mathcal{V}}_h = \{x_i, \tilde{\mathcal{V}}_i\}_{i=1}^N$ be an almost uniform Voronoï-Delaunay decomposition on \mathbb{S}^2 . Consider $\overline{\mathcal{F}}_{ij}$ as the discrete flux defined in (4.2). Then, for the solution $u_h \in \tilde{U}_h$ of problem (4.12), there are positive constants C_0 and C_1 such that*

$$\begin{aligned} \overline{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(v_h)) &\leq C_0 |u_h|_{1,h} |v_h|_{1,h}, \quad \text{for each } v_h \in \tilde{U}_h, \\ \overline{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(u_h)) &\geq C_1 |u_h|_{1,h}^2. \end{aligned}$$

Here $|\cdot|_{1,h}$ denotes the discrete seminorm with $p = 2$.

The following result establishes an estimate of the stability of scheme (4.4), which is an immediate consequence of the proposition above.

Proposition 4.6 *Let $f \in L^2(\mathbb{S}^2)$ satisfy the compatibility condition (2.1). A unique approximate solution $u_h \in \tilde{U}_h$ of the discrete problem (4.12) satisfies,*

$$|u_h|_{1,h} \leq C \|f\|_{L^2(\mathbb{S}^2)}, \quad (4.13)$$

where C is a positive constant independent of parameter h .

4.3 Geometric error estimates

In this subsection, we present two bounds concerning the geometric perturbation errors in the bilinear forms. We begin with the following lemma.

Lemma 4.7 Let $\tilde{\mathcal{A}}_h(\cdot, \cdot)$ and $\bar{\mathcal{A}}_h(\cdot, \cdot)$ be the total flux bilinear forms defined in (4.10) and (4.11) respectively. Assume that $u_h \in \tilde{U}_h$ is the unique solution to the discrete problem (4.12). Then, for each $v_h \in \tilde{U}_h$ and $p > 1$ with $1/p + 1/q = 1$, there exists a positive constant C , independent of h , such that

$$|\tilde{\mathcal{A}}_h(u_h, \tilde{I}_h(v_h)) - \bar{\mathcal{A}}_h(u_h, \tilde{I}_h(v_h))| \leq Ch^2 \|\nabla_s u_h\|_{L^p(\mathbb{S}^2)} \|\nabla_s v_h\|_{L^q(\mathbb{S}^2)}.$$

Proof This estimate was established for $p = q = 2$ in [20, Lemma 4, pp. 1686]. The same proof applies to show the estimates for general p and q using a Hölder inequality and norm equivalence from Proposition 4.4. \square

We establish the following technical lemma.

Lemma 4.8 Let $x \in \mathbb{S}^2$ and $w_h \in \tilde{U}_h$, then there exists a positive constant C , independent of h , such that

$$|\Delta_s w_h(x)| \leq Ch \|\nabla_s w_h\|. \quad (4.14)$$

Proof Take a planar triangle $T_{ijk} \in \mathcal{T}_h$ and its radial projection $\tilde{T}_{ijk} = \mathcal{P}(T_{ijk})$, using $x = \mathcal{P}(x^*) \in \tilde{T}_{ijk}$, for $x^* \in T_{ijk}$. Without loss of generality, to simplify calculations, assume this triangle lies on a plane of constant third coordinate x_3^* so that \bar{x}_3^* of $x^* = (x_1^*, x_2^*, \bar{x}_3^*) \in T_{ijk}$, with fixed \bar{x}_3^* , can fully define the planar triangle in terms of the variables (x_1^*, x_2^*) .¹

By construction, we have $1 - Ch^2 < \|x^*\| \leq 1$, for h small enough. Let $w_h : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}$ be linear (affine) on T_{ijk} , that is, $w_h(x_1^*, x_2^*, \bar{x}_3^*) = ax_1^* + bx_2^* + c\bar{x}_3^* + d = ax_1^* + bx_2^* + e$, where $e = c\bar{x}_3^* + d$ is constant and $a, b, c, d, e \in \mathbb{R}$. In spherical coordinates, we get that

$$x_1^* = r \cos \theta \sin \phi, \quad x_2^* = r \sin \theta \sin \phi, \quad x_3^* = r \cos \phi, \quad \text{with } \theta \in [-\pi, \pi], \quad \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}],$$

where

$$r = \|x^*\|, \quad \theta = \arctan\left(\frac{x_2^*}{x_1^*}\right), \quad \phi = \arccos\left(\frac{x_3^*}{r}\right).$$

Then, for \bar{x}_3^* fixed, we can write,

$$w_h(x_1^*, x_2^*, \bar{x}_3^*) = ax_1^* + bx_2^* + e = a\bar{x}_3^* \cos \theta \tan \phi + b\bar{x}_3^* \sin \theta \tan \phi + e.$$

Notice that $w_h(r, \theta, \phi)$ is constant for given fixed θ and ϕ , and therefore is invariant with respect to r on T_{ijk} , thus $w_h \in \tilde{U}_h$. Now,

$$\nabla w_h(r, \theta, \phi) = \vec{r} \frac{\partial}{\partial r} w_h(r, \theta, \phi) + \vec{\theta} \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta} w_h(r, \theta, \phi) + \vec{\phi} \frac{1}{r} \frac{\partial}{\partial \phi} w_h(r, \theta, \phi),$$

¹ Any other planar triangle of the grid can be obtained in this representation by rotations of the sphere.

where $\{\vec{r}, \vec{\theta}, \vec{\phi}\}$ are the local orthogonal unit vectors in the directions of increasing r , θ , and ϕ . Observe that $\partial_r w_h(r, \theta, \phi) = 0$, so

$$\nabla w_h(r, \theta, \phi) = \vec{\theta} \frac{1}{r \sin \phi} \tan \phi (-a\bar{x}_3^* \sin \theta + b\bar{x}_3^* \cos \theta) + \vec{\phi} \frac{1}{r} \sec^2 \phi (a\bar{x}_3^* \cos \theta + b\bar{x}_3^* \sin \theta).$$

By a similar calculation, we can find that

$$\begin{aligned} \Delta w_h(r, \theta, \phi) &= \frac{1}{r^2} \partial_\phi^2 w_h + \frac{1}{r^2 \sin^2 \phi} \partial_\theta^2 w_h + \frac{1}{r^2} \frac{\cos \phi}{\sin \phi} \partial_\phi w_h \\ &= \frac{2}{r^2} \tan \phi \sec^2 \phi (a\bar{x}_3^* \cos \theta + b\bar{x}_3^* \sin \theta) - \frac{1}{r^2 \sin^2 \phi} \tan \phi (a\bar{x}_3^* \cos \theta + b\bar{x}_3^* \sin \theta) \\ &\quad + \frac{1}{r^2} \frac{\cos \phi}{\sin \phi} \sec^2 \phi (a\bar{x}_3^* \cos \theta + b\bar{x}_3^* \sin \theta) \\ &= \frac{2}{r^2} \tan \phi \sec^2 \phi (a\bar{x}_3^* \cos \theta + b\bar{x}_3^* \sin \theta) - \frac{1}{r^2 \sin^2 \phi} \tan \phi (a\bar{x}_3^* \cos \theta + b\bar{x}_3^* \sin \theta) \\ &\quad + \frac{1}{r^2} \frac{1}{\sin^2 \phi} \frac{\sin \phi}{\cos \phi} (a\bar{x}_3^* \cos \theta + b\bar{x}_3^* \sin \theta) \\ &= \frac{2}{r} \tan \phi (\nabla w_h(r, \theta, \phi) \cdot \vec{\phi}). \end{aligned}$$

Then, for $\phi \in [-\frac{h}{2}, \frac{h}{2}]$, and r is in the shell, i.e., $1 - Ch^2 < r < 1 + Ch^2$, we have that

$$|\Delta w_h(r, \theta, \phi)| \leq \left| \frac{2}{1 - Ch^2} \tan \frac{h}{2} \right| \|\nabla w_h(r, \theta, \phi)\|,$$

Observe that

$$\frac{1}{1 - Ch^2} \tan \frac{h}{2} = \left(1 + Ch^2 + C^2 h^4 + \dots\right) \left(\frac{h}{2} + \frac{1}{3} \left(\frac{h}{2}\right)^3 + \dots\right) = \frac{h}{2} + \mathcal{O}(h^3).$$

Therefore,

$$|\Delta w_h(r, \theta, \phi)| \leq Ch \|\nabla w_h(r, \theta, \phi)\|.$$

Taking $r = 1$, and using Proposition 3.4, we arrive at (4.14). \square

Lemma 4.9 Assume $p > 1$ such that $1/p + 1/q = 1$. Let $\tilde{\mathcal{V}}_h = \{x_i, \tilde{\mathcal{V}}_i\}_{i=1}^N$ be an almost uniform Voronoï-Delaunay decomposition on \mathbb{S}^2 . Let $\mathcal{A}(\cdot, \cdot)$ and $\tilde{\mathcal{A}}_h(\cdot, \cdot)$ be the bilinear forms defined in (2.3) and (4.10) respectively. Also, assume that $u_h \in \tilde{U}_h$ is the unique solution of problem (4.12). Then, there exists a positive constant C , independent of h , such that

$$|\mathcal{A}(u_h, v_h) - \tilde{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(v_h))| \leq Ch^2 \|\nabla_s u_h\|_{L^p(\mathbb{S}^2)} \|\nabla_s v_h\|_{L^q(\mathbb{S}^2)}, \quad (4.15)$$

for each $v_h \in \tilde{U}_h$.

Proof Given $u_h \in \tilde{U}_h$, then $u_h|_{\tilde{T}_{ijk}} \in W^{2,p}(\tilde{T}_{ijk})$ for each $\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h$. Multiplying $-\Delta_s u_h$ by $v_h \in \tilde{U}_h$ and by using Gauss Theorem, we have

$$\begin{aligned} - \int_{\tilde{T}_{ijk}} \Delta_s u_h(x) v_h(x) ds(x) &= \int_{\tilde{T}_{ijk}} \nabla_s u_h(x) \cdot \nabla_s v_h(x) ds(x) \\ &\quad - \int_{\partial \tilde{T}_{ijk}} \nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{T}_{ijk}} v_h(x) d\gamma(x). \end{aligned}$$

From definition of $\mathcal{A}(\cdot, \cdot)$ and summing up all $\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h$, we get

$$\mathcal{A}(u_h, v_h) = \sum_{\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h} \left[\int_{\tilde{T}_{ijk}} -\Delta_s u_h(x) v_h(x) ds(x) + \int_{\partial \tilde{T}_{ijk}} \nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{T}_{ijk}} v_h(x) d\gamma(x) \right]. \quad (4.16)$$

From Lemma 4.3, we consider again the spherical polygonal regions \tilde{Q}_i, \tilde{Q}_j and \tilde{Q}_k made of the intersection of the geodesic triangle \tilde{T}_{ijk} with the three Voronoï cells associated to each vertex of the triangle, i.e., $\tilde{Q}_n = \tilde{V}_n \cap \tilde{T}_{ijk}$, $n \in \{i, j, k\}$, with boundaries

$$\partial \tilde{Q}_n = (\partial \tilde{V}_n \cap \tilde{T}_{ijk}) \cup (\partial \tilde{T}_{ijk} \cap \tilde{V}_n), \quad n \in \{i, j, k\}.$$

Now, multiplying $-\Delta_s u_h$ by $\tilde{I}_h(v_h) \in \tilde{V}_h$ and integrating over \tilde{T}_{ijk} , follows that

$$\begin{aligned} - \int_{\tilde{T}_{ijk}} \Delta_s u_h(x) \tilde{I}_h(v_h)(x) ds(x) &= \sum_{n=i,j,k} \int_{\tilde{Q}_n} \nabla_s u_h(x) \cdot \nabla_s \tilde{I}_h(v_h)(x) ds(x) \\ &\quad - \sum_{n=i,j,k} \int_{\partial \tilde{Q}_n} \nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{Q}_n} \tilde{I}_h(v_h)(x) d\gamma(x) \\ &= - \sum_{n=i,j,k} \int_{\partial \tilde{Q}_n} \nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{Q}_n} \tilde{I}_h(v_h)(x) d\gamma(x). \end{aligned}$$

Rearranging the boundary $\partial \tilde{Q}_n$ with $n \in \{i, j, k\}$, we have

$$\begin{aligned} - \int_{\tilde{T}_{ijk}} \Delta_s u_h(x) \tilde{I}_h(v_h)(x) ds(x) &= - \int_{\partial \tilde{T}_{ijk}} \nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{T}_{ijk}} \tilde{I}_h(v_h)(x) d\gamma(x) \\ &\quad - \sum_{n=i,j,k} \int_{\partial \tilde{V}_n \cap \tilde{T}_{ijk}} \nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{V}_n} \tilde{I}_h(v_h)(x) d\gamma(x). \end{aligned}$$

Now, summing up all geodesic triangles of $\tilde{\mathcal{T}}_h$ and using duality principle, i.e., each edge of $\tilde{\mathcal{T}}_{ijk}$ intersects to a unique dual Voronoï edge, we get

$$\begin{aligned} - \sum_{\tilde{\mathcal{T}}_{ijk} \in \tilde{\mathcal{T}}_h} \int_{\tilde{\mathcal{T}}_{ijk}} \Delta_s u_h(x) \tilde{\mathcal{I}}_h(v_h)(x) ds(x) &= - \sum_{\tilde{\mathcal{T}}_{ijk} \in \tilde{\mathcal{T}}_h} \int_{\partial \tilde{\mathcal{T}}_{ijk}} \nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{\mathcal{T}}_{ijk}} \tilde{\mathcal{I}}_h(v_h)(x) d\gamma(x) \\ &\quad - \sum_{i=1}^N \int_{\partial \tilde{\mathcal{V}}_i} \nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{\mathcal{V}}_i} \tilde{\mathcal{I}}_h(v_h)(x) d\gamma(x). \end{aligned}$$

The last term above on the right-hand side is the bilinear form $\tilde{\mathcal{A}}_h(\cdot, \cdot)$. Therefore

$$\tilde{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(v_h)) = \sum_{\tilde{\mathcal{T}}_{ijk} \in \tilde{\mathcal{T}}_h} \left[- \int_{\tilde{\mathcal{T}}_{ijk}} \Delta_s u_h(x) \tilde{\mathcal{I}}_h(v_h)(x) ds(x) + \int_{\partial \tilde{\mathcal{T}}_{ijk}} \nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{\mathcal{T}}_{ijk}} \tilde{\mathcal{I}}_h(v_h)(x) d\gamma(x) \right]. \quad (4.17)$$

Then, we subtract (4.17) from (4.16) to obtain

$$\begin{aligned} |\mathcal{A}(u_h, v_h) - \tilde{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(v_h))| &\leq \left| \sum_{\tilde{\mathcal{T}}_{ijk} \in \tilde{\mathcal{T}}_h} \int_{\tilde{\mathcal{T}}_{ijk}} \Delta_s u_h(x) [v_h(x) - \tilde{\mathcal{I}}_h(v_h)(x)] ds(x) \right| \\ &\quad + \left| \sum_{\tilde{\mathcal{T}}_{ijk} \in \tilde{\mathcal{T}}_h} \int_{\partial \tilde{\mathcal{T}}_{ijk}} \nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{\mathcal{T}}_{ijk}} [v_h(x) - \tilde{\mathcal{I}}_h(v_h)(x)] d\gamma(x) \right|. \end{aligned}$$

For each $\tilde{v}_{ij} \in \partial \tilde{\mathcal{T}}_{ijk}$ the function $\nabla_s u_h(x) \cdot \vec{n}_{x, \tilde{\mathcal{T}}_{ijk}} [v_h(x) - \tilde{\mathcal{I}}_h(v_h)(x)]$ is anti-symmetric with respect to the edge's midpoint, and therefore as shown in Lemma 4.3 its integral along the edge vanishes. It follows that

$$|\mathcal{A}(u_h, v_h) - \tilde{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(v_h))| \leq \left| \sum_{\tilde{\mathcal{T}}_{ijk} \in \tilde{\mathcal{T}}_h} \int_{\tilde{\mathcal{T}}_{ijk}} \Delta_s u_h(x) [v_h(x) - \tilde{\mathcal{I}}_h(v_h)(x)] ds(x) \right|.$$

Finally, invoking Lemma 4.8, the Hölder inequality, Proposition 3.5 and Lemma 4.3, we arrive to

$$\begin{aligned} |\mathcal{A}(u_h, v_h) - \tilde{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(v_h))| &\leq \sum_{\tilde{\mathcal{T}}_{ijk} \in \tilde{\mathcal{T}}_h} \int_{\tilde{\mathcal{T}}_{ijk}} |\Delta_s u_h(x)| |v_h(x) - \tilde{\mathcal{I}}_h(v_h)(x)| ds(x) \\ &\leq Ch \sum_{\tilde{\mathcal{T}}_{ijk} \in \tilde{\mathcal{T}}_h} \int_{\tilde{\mathcal{T}}_{ijk}} \|\nabla_s u_h\| |v_h(x) - \tilde{\mathcal{I}}_h(v_h)(x)| ds(x) \\ &\leq Ch^2 \|\nabla_s u_h\|_{L^p(\mathbb{S}^2)} \|\nabla_s v_h\|_{L^q(\mathbb{S}^2)}. \end{aligned}$$

This finishes the proof. \square

Lemma 4.10 Assume that $f \in L^p(\mathbb{S}^2)$, with $p > 1$ satisfies the compatibility condition (2.1). Then, for each $v_h \in \tilde{U}_h$, with $1/p + 1/q = 1$ there exists a positive constant C , independent of h , such that

$$|(f, v_h) - (f, \tilde{I}_h(v_h))| \leq Ch \|f\|_{L^p(\mathbb{S}^2)} \|\nabla_s v_h\|_{L^q(\mathbb{S}^2)}, \quad (4.18)$$

Proof By applying the Hölder's inequality with $1/p + 1/q = 1$ and using Lemma 4.3, we obtain

$$\begin{aligned} |(f, v_h) - (f, \tilde{I}_h(v_h))| &= \left| \sum_{\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h} \int_{\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h} f(x) [v_h(x) - \tilde{I}_h(v_h)(x)] ds(x) \right| \\ &\leq Ch \sum_{\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h} \|f\|_{L^p(\tilde{T}_{ijk})} \|\nabla_s v_h\|_{L^q(\tilde{T}_{ijk})} \\ &\leq Ch \|f\|_{L^p(\mathbb{S}^2)} \|\nabla_s v_h\|_{L^q(\mathbb{S}^2)}. \end{aligned}$$

Therefore, we get (4.18). \square

5 Error analysis

In this section, we establish the estimates of convergence order of the approximate solutions of FVM in classical H^1 , L^2 -norm and max-norm using the framework of [20, 22, 34] for Voronoï-Delaunay decomposition on \mathbb{S}^2 and highlight that the estimated convergence rates depend on the position of vertices of the geometric setting.

5.1 Classical H^1 and L^2 estimates

The following result provides an error estimate of the finite volume solution u_h in the H^1 and L^2 -norms with minimal regularity assumptions for the exact solution u and is valid for Voronoï-Delaunay decompositions in general on \mathbb{S}^2 .

Theorem 5.1 Let $\tilde{\mathcal{V}}_h = \{x_i, \tilde{\mathcal{V}}_i\}_{i=1}^N$ be an almost uniform Voronoï-Delaunay decomposition on \mathbb{S}^2 . Assume that $f \in L^2(\mathbb{S}^2)$ satisfies (2.1) and the unique solution u of (2.4) belongs to $H^2(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$. Let $\bar{\mathcal{F}}_{ij}$ be the discrete flux defined in (4.2), such that the discrete problem (4.12) has a unique solution $u_h \in \tilde{U}_h$. Then, there exists a positive constant C , independent of h , such that

$$\|\nabla_s \varepsilon_h\|_{L^2(\mathbb{S}^2)} \leq Ch (\|u\|_{H^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)}), \quad (5.1a)$$

$$\|\varepsilon_h\|_{L^2(\mathbb{S}^2)} \leq Ch \|f\|_{L^2(\mathbb{S}^2)} + Ch^2 \|u\|_{H^2(\mathbb{S}^2)}, \quad (5.1b)$$

where $\varepsilon_h = u - u_h$.

Proof Firstly, for (5.1a) we consider the coercivity of $\mathcal{A}(\cdot, \cdot)$, and define $\varphi_h = \tilde{\Pi}_h(u) - u_h$. Then by Proposition 4.1 and triangular inequality, we get

$$\begin{aligned} \|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)}^2 &\leq |\mathcal{A}(u - u_h, u - u_h)| \\ &\leq |\mathcal{A}(u - u_h, u - \tilde{\Pi}_h(u))| + |\mathcal{A}(u, \varphi_h) - \mathcal{A}(u_h, \varphi_h)| \\ &\leq |\mathcal{A}(u - u_h, u - \tilde{\Pi}_h(u))| + |\mathcal{A}(u, \varphi_h) - \bar{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(\varphi_h))| \\ &\quad + |\bar{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(\varphi_h)) - \tilde{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(\varphi_h))| \\ &\quad + |\tilde{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(\varphi_h)) - \mathcal{A}(u_h, \varphi_h)| \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned} \quad (5.2)$$

where the hidden constant in “ \leq ” comes from the coercivity of $\mathcal{A}(\cdot, \cdot)$. For I_1 , applying the continuity of $\mathcal{A}(\cdot, \cdot)$ and Proposition 4.1, we have

$$\begin{aligned} I_1 = |\mathcal{A}(u - u_h, u - \tilde{\Pi}_h(u))| &\leq C \|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)} \|\nabla_s(u - \tilde{\Pi}_h(u))\|_{L^2(\mathbb{S}^2)} \\ &\leq Ch \|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)} \|u\|_{H^2(\mathbb{S}^2)}. \end{aligned} \quad (5.3)$$

For I_2 , from right-hand sides of variational problems (2.4) and (4.12) and putting $p = q = 2$ in Lemma 4.10, we find

$$I_2 = |\mathcal{A}(u, \varphi_h) - \bar{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(\varphi_h))| = |(f, \varphi_h - \tilde{\Gamma}_h(\varphi_h))| \leq Ch \|f\|_{L^2(\mathbb{S}^2)} \|\nabla_s \varphi_h\|_{L^2(\mathbb{S}^2)}.$$

About I_3 , by Lemma 4.7, equation (4.13) and Proposition 4.4 follows that

$$\begin{aligned} I_3 = |\bar{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(\varphi_h)) - \tilde{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(\varphi_h))| &\leq Ch^2 \|\nabla_s u_h\|_{L^2(\mathbb{S}^2)} \|\nabla_s \varphi_h\|_{L^2(\mathbb{S}^2)} \\ &\leq Ch^2 \|f\|_{L^2(\mathbb{S}^2)} \|\nabla_s \varphi_h\|_{L^2(\mathbb{S}^2)}. \end{aligned} \quad (5.4)$$

Similarly, for I_4 we use Lemma 4.9 to get

$$\begin{aligned} I_4 = |\tilde{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(\varphi_h)) - \mathcal{A}(u_h, \varphi_h)| &\leq Ch^2 \|\nabla_s u_h\|_{L^2(\mathbb{S}^2)} \|\nabla_s \varphi_h\|_{L^2(\mathbb{S}^2)} \\ &\leq Ch^2 \|f\|_{L^2(\mathbb{S}^2)} \|\nabla_s \varphi_h\|_{L^2(\mathbb{S}^2)}. \end{aligned} \quad (5.5)$$

Gathering (5.2) and inequalities (5.3)–(5.5), we have an expression as

$$\begin{aligned} \|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)}^2 &\leq Ch \|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)} \|u\|_{H^2(\mathbb{S}^2)} + Ch \|f\|_{L^2(\mathbb{S}^2)} \|\nabla_s \varphi_h\|_{L^2(\mathbb{S}^2)} \\ &\quad + Ch^2 \|f\|_{L^2(\mathbb{S}^2)} \|\nabla_s \varphi_h\|_{L^2(\mathbb{S}^2)}. \end{aligned} \quad (5.6)$$

Using Young's and triangle inequalities, and Proposition 4.1, we find

$$\begin{aligned} \|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)}^2 &\leq \epsilon \left(\|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)}^2 + \|\nabla_s \varphi_h\|_{L^2(\mathbb{S}^2)}^2 \right) + h^2 \left(\|u\|_{H^2(\mathbb{S}^2)}^2 + \|f\|_{L^2(\mathbb{S}^2)}^2 \right) \\ &\quad + h^4 \|f\|_{L^2(\mathbb{S}^2)}^2 \\ &\leq \epsilon \left(\|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)}^2 + Ch^2 \|u\|_{H^2(\mathbb{S}^2)}^2 \right) + h^2 \left(\|u\|_{H^2(\mathbb{S}^2)}^2 + \|f\|_{L^2(\mathbb{S}^2)}^2 \right) \\ &\quad + h^4 \|f\|_{L^2(\mathbb{S}^2)}^2. \end{aligned}$$

Taking ϵ small enough, we arrive to

$$\|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)}^2 \leq h^2 \left(\|u\|_{H^2(\mathbb{S}^2)}^2 + \|f\|_{L^2(\mathbb{S}^2)}^2 \right) + h^4 \|f\|_{L^2(\mathbb{S}^2)}^2.$$

Finally, for h small enough, we obtain (5.1a).

In order to prove (5.1b), we derive an error estimate following the classical duality argument: for $u - u_h \in H_0^1(\mathbb{S}^2)$, from (2.4), there exists a unique weak solution $w \in H^2(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$ satisfying

$$\mathcal{A}(v, w) = (v, u - u_h), \quad \text{for all } v \in H_0^1(\mathbb{S}^2).$$

Taking $v = u - u_h$ and by using the regularity estimate (2.6), we have

$$\|w\|_{H^2(\mathbb{S}^2)} \leq C \|u - u_h\|_{L^2(\mathbb{S}^2)}. \quad (5.7)$$

Now, assume that $w_h = \tilde{\Pi}_h(w) \in \tilde{U}_h$, we obtain

$$\begin{aligned} \|u - u_h\|_{L^2(\mathbb{S}^2)}^2 &= (u - u_h, u - u_h) \leq |\mathcal{A}(u - u_h, w)| \\ &= |\mathcal{A}(u - u_h, w - w_h) + \mathcal{A}(u, w_h) - \mathcal{A}(u_h, w_h)| \\ &\leq |\mathcal{A}(u - u_h, w - w_h)| + |\mathcal{A}(u, w_h) - \bar{\mathcal{A}}_h(u_h, \tilde{\mathbf{I}}_h(w_h))| \\ &\quad + |\bar{\mathcal{A}}_h(u_h, \tilde{\mathbf{I}}_h(w_h)) - \tilde{\mathcal{A}}_h(u_h, \tilde{\mathbf{I}}_h(w_h))| \\ &\quad + |\tilde{\mathcal{A}}_h(u_h, \tilde{\mathbf{I}}_h(w_h)) - \mathcal{A}(u_h, w_h)| \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

About I_1 , by equation (5.1a), Proposition 4.1 and inequality (5.7), we have

$$\begin{aligned} I_1 &= |\mathcal{A}(u - u_h, w - w_h)| \leq C \|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)} \|\nabla_s(w - w_h)\|_{L^2(\mathbb{S}^2)} \\ &\leq Ch^2 \left(\|u\|_{H^2(\mathbb{S}^2)}^2 + \|f\|_{L^2(\mathbb{S}^2)}^2 \right) \|u - u_h\|_{L^2(\mathbb{S}^2)}. \end{aligned} \quad (5.8)$$

For I_2 , by Lemma 4.10 and expression (5.7), we have

$$I_2 = |\mathcal{A}(u, w_h) - \bar{\mathcal{A}}_h(u_h, \tilde{\mathbf{I}}_h(w_h))| \leq Ch \|f\|_{L^2(\mathbb{S}^2)} \|u - u_h\|_{L^2(\mathbb{S}^2)}. \quad (5.9)$$

Analogously for I_3 , using Lemma 4.7, equations (4.13) and (5.7) follows that

$$I_3 = |\overline{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(w_h)) - \tilde{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(w_h))| \leq Ch^2 \|f\|_{L^2(\mathbb{S}^2)} \|u - u_h\|_{L^2(\mathbb{S}^2)}. \quad (5.10)$$

Finally, by Lemma 4.9 along with estimates (4.13) and (5.7) yields the expression

$$I_4 = |\tilde{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(w_h)) - \mathcal{A}(u_h, w_h)| \leq Ch^2 \|f\|_{L^2(\mathbb{S}^2)} \|u - u_h\|_{L^2(\mathbb{S}^2)}. \quad (5.11)$$

Combining (5.8)–(5.11) with $h > 0$ small enough, we have

$$\|u - u_h\|_{L^2(\mathbb{S}^2)}^2 \leq (h\|f\|_{L^2(\mathbb{S}^2)} + h^2\|f\|_{L^2(\mathbb{S}^2)} + h^2\|u\|_{H^2(\mathbb{S}^2)}) \|u - u_h\|_{L^2(\mathbb{S}^2)}.$$

Dividing by $\|u - u_h\|_{L^2(\mathbb{S}^2)}$ encloses the proof of the theorem. \square

Remark 5.2 Notice that the convergence rate in the L^2 -norm is lower than that obtained in FEM. As we will see in the numerical experiment, the errors of the finite volume solutions behave better than the estimates given above. Meanwhile, no known standard method exists to increase the convergence estimate in the L^2 -norm by using Voronoï-Delaunay decomposition in general.

The dominant term in the bounds of Theorem 5.1 is given by Lemma 4.10. One way to gain extra power in the convergence rate is to use the SCVT optimizations in the standard decomposition. A quadratic order estimate was reported by [20]. Still, we will illustrate below that a decrease in regularity in the exact solution is possible using the techniques investigated by [22].

The following lemma is a simplified version of a result given by [20, Lemma 1, pp. 1682], assuming that the density function ρ is equal to 1.

Lemma 5.3 [20] *Let $\tilde{\mathcal{V}}_h = \{x_i, \tilde{\mathbf{V}}_i\}_{i=1}^N$ be an almost uniform Spherical Centroidal Voronoï-Delaunay decomposition (SCVT) on \mathbb{S}^2 . Then, for any $w \in H^2(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$, there exists a positive constant C , independent of h , such that*

$$\left| \int_{\tilde{\mathbf{V}}_i} [w(x) - \tilde{\Pi}_h^*(w)(x)] ds(x) \right| \leq Ch^2 m_a(\tilde{\mathbf{V}}_i)^{1/2} \|w\|_{H^2(\tilde{\mathbf{V}}_i)}.$$

In light of the lemma above, we have the following result.

Lemma 5.4 *Let $f \in H_0^1(\mathbb{S}^2)$ be a function that satisfies the compatibility condition (2.1) and $\tilde{\mathcal{V}}_h = \{x_i, \tilde{\mathbf{V}}_i\}_{i=1}^N$ be an almost uniform Spherical Centroidal Voronoï-Delaunay decomposition (SCVT) on \mathbb{S}^2 . Then, for each $w \in H^2(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$, there exists a positive constant C , independent of h , such that*

$$(f, \tilde{\Pi}_h(w) - \tilde{\Pi}_h^*(w)) \leq Ch^2 \|f\|_{H^1(\mathbb{S}^2)} \|w\|_{H^2(\mathbb{S}^2)},$$

where $\tilde{\Pi}_h$ and $\tilde{\Pi}_h^*$ are the interpolation operator on spaces \tilde{U}_h and \tilde{V}_h respectively.

Proof By definition of the inner product, we have

$$\begin{aligned} (f, \tilde{\Pi}_h(w) - \tilde{\Pi}_h^*(w)) &= \sum_{\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h} \int_{\tilde{T}_{ijk}} f(x) [\tilde{\Pi}_h(w)(x) - w(x)] ds(x) \\ &\quad + \sum_{i=1}^N \int_{\tilde{V}_i} [f(x) - P_h(f)] [w(x) - \tilde{\Pi}_h^*(w)(x)] ds(x) \\ &\quad + \sum_{i=1}^N \int_{\tilde{V}_i} P_h(f) [w(x) - \tilde{\Pi}_h^*(w)(x)] ds(x) \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where $P_h(f)$ denotes the L^2 -projection of the function f on \tilde{V}_h . For I_1 , by using Cauchy-Schwarz inequality and Proposition 4.1, we obtain

$$\begin{aligned} I_1 &\leq \sum_{\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h} \left(\int_{\tilde{T}_{ijk}} |f(x)|^2 ds(x) \right)^{1/2} \left(\int_{\tilde{T}_{ijk}} |\tilde{\Pi}_h(w)(x) - w(x)|^2 ds(x) \right)^{1/2} \\ &\leq Ch^2 \|f\|_{L^2(\mathbb{S}^2)} \|w\|_{H^2(\mathbb{S}^2)}. \end{aligned} \quad (5.12)$$

Analogously, for I_2 ,

$$\begin{aligned} I_2 &\leq \sum_{i=1}^N \left(\int_{\tilde{V}_i} |f(x) - P_h(f)(x)|^2 ds(x) \right)^{1/2} \left(\int_{\tilde{V}_i} |w(x) - \tilde{\Pi}_h^*(w)(x)|^2 ds(x) \right)^{1/2} \\ &\leq Ch \sum_{i=1}^N \|\nabla_s f\|_{L^2(\tilde{V}_i)} \|w - \tilde{\Pi}_h^*(w)\|_{L^2(\tilde{V}_i)} \leq Ch^2 \|f\|_{H^1(\mathbb{S}^2)} \|w\|_{H^2(\mathbb{S}^2)}. \end{aligned} \quad (5.13)$$

Finally, for I_3 , using Lemma 5.3 and Proposition 4.4, we have that

$$\begin{aligned} I_3 &\leq Ch^2 \sum_{i=1}^N |P_h(f)|_{\tilde{V}_i} m_a(\tilde{V}_i)^{1/2} \|w\|_{H^2(\tilde{V}_i)} \\ &\leq Ch^2 \left(\sum_{i=1}^N m_a(\tilde{V}_i) |P_h(f)|_{\tilde{V}_i}^2 \right)^{1/2} \left(\sum_{i=1}^N \|w\|_{H^2(\tilde{V}_i)}^2 \right)^{1/2} \\ &\leq Ch^2 \|f\|_{L^2(\mathbb{S}^2)} \|w\|_{H^2(\mathbb{S}^2)}. \end{aligned} \quad (5.14)$$

Combining (5.12)–(5.14) we obtain the result. \square

We will show below a subtle modification of the proof given in [20], assuming that the exact solution $u \in H^2(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$ and source term $f \in H_0^1(\mathbb{S}^2)$.

Theorem 5.5 Let $\tilde{\mathcal{V}}_h = \{\mathbf{x}_i, \tilde{\mathcal{V}}_i\}_{i=1}^N$ be an almost uniform Spherical Centroidal Voronoï-Delaunay decomposition (SCVT) on \mathbb{S}^2 . Assume that $f \in H_0^1(\mathbb{S}^2)$ and that $u \in H^2(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$ is the unique solution of (2.2). Let $\bar{\mathcal{F}}_{ij}$ be the discrete flux defined in (4.2), such that the discrete problem (4.12) has a unique solution $u_h \in \tilde{\mathcal{U}}_h$. Then, there exists a positive constant C , independent of h , such that

$$\|\varepsilon_h\|_{L^2(\mathbb{S}^2)} \leq Ch^2 (\|u\|_{H^2(\mathbb{S}^2)} + \|f\|_{H^1(\mathbb{S}^2)}),$$

where $\varepsilon_h = u - u_h$.

Proof The proof is analogous to (5.1b) in Theorem 5.1, now using Lemma 5.4 into (5.9) to guarantee an extra power in the term I_2 , i.e.,

$$I_2 = |(f, w_h - \tilde{\mathcal{I}}_h(w_h))| \leq Ch^2 \|f\|_{H^1(\mathbb{S}^2)} \|w\|_{H^2(\mathbb{S}^2)} \leq Ch^2 \|f\|_{H^1(\mathbb{S}^2)} \|u - u_h\|_{L^2(\mathbb{S}^2)},$$

where $w_h = \tilde{\Pi}_h(w)$. Therefore, the quadratic order is shown and completes the proof of the theorem. \square

5.2 Pointwise error estimates

In this subsection, we find error estimates in the maximum norm for problem (2.2). We shall use the variational formulation (2.3) for regularized Green's functions. We consider some properties of these auxiliary functions along with punctual estimates of FEM on surfaces defined by [13, 38].

5.2.1 Regularity properties of Green's functions

The lemma below has fundamental properties of the regularized Green's functions on \mathbb{S}^2 . The proof is detailed in [1, Theorem 4.13, pp. 108].

Lemma 5.6 [1] *There exists $G(\mathbf{x}, \mathbf{y})$, a Green's function for the Laplacian on \mathbb{S}^2 satisfying, for each $u \in C^2(\mathbb{S}^2)$ and $\mathbf{x}, \mathbf{y} \in \mathbb{S}^2$ with $\mathbf{x} \neq \mathbf{y}$,*

$$u(\mathbf{y}) = \frac{1}{m_a(\mathbb{S}^2)} \int_{\mathbb{S}^2} u(\mathbf{x}) ds(\mathbf{x}) - \int_{\mathbb{S}^2} G(\mathbf{x}, \mathbf{y}) \Delta_s u(\mathbf{x}) ds(\mathbf{x}),$$

and there exist positive constants C_0, C_1 and C_2 such that

$$|G(\mathbf{x}, \mathbf{y})| \leq C_0(1 + |\ln d(\mathbf{x}, \mathbf{y})|),$$

$$|\nabla_s G(\mathbf{x}, \mathbf{y})| \leq C_1 \frac{1}{|d(\mathbf{x}, \mathbf{y})|}, \quad |\Delta_s G(\mathbf{x}, \mathbf{y})| \leq C_2 \frac{1}{|d(\mathbf{x}, \mathbf{y})|^2},$$

where ∇_s and Δ_s denote the tangential gradient and Laplacian acting on a function of \mathbf{x} . Finally, $G(\mathbf{x}, \mathbf{y})$ satisfies

$$\int_{\mathbb{S}^2} G(\mathbf{x}, \mathbf{y}) ds(\mathbf{x}) = 0. \quad (5.15)$$

From now on, we denote by $G^y(x)$ the Green's function $G(x, y)$. Further, we consider the notation g^y to refer to the discrete Green's function defined as

$$g^y(x) := \int_{\mathbb{S}^2} G^x(z) \delta^y(z) ds(z), \quad (5.16)$$

where $\delta^y(z)$ represents the discrete Dirac delta function at the point y .

We now present a proposition by Demlow [13, Proposition 2.8, pp. 813], as it will be used further in the analysis.

Proposition 5.7 [13] *Consider $v_h \in \tilde{U}_h$ and fix $y \in \tilde{T}_{ijk} \subset \mathbb{S}^2$. Let $\vec{n}_{y, \tilde{T}_{ijk}}$ be a unit vector on the tangent plane $\mathbb{T}_{y, \mathbb{S}^2}$ at y . Then, there exist δ^y and $\delta^{y, \Omega}$, both independent of v_h , such that for some positive constant C ,*

$$\|\delta^y\|_{W^{m,p}(\tilde{T}_{ijk})} + \|\delta^{y, \Omega}\|_{W^{m,p}(\tilde{T}_{ijk})} \leq Ch^{-m-2\left(\frac{p-1}{p}\right)},$$

for $m \in \{0, 1\}$ and $1 \leq p \leq \infty$. Further, there exists positive generic constants C_0 and C_1 , such that

$$\begin{aligned} |v_h(y)| &\leq C_0 \left| \int_{\tilde{T}_{ijk}} \delta^y(x) v_h(x) ds(x) \right|, \\ \left| \nabla_s v_h(y) \cdot \vec{n}_{y, \tilde{T}_{ijk}} \right| &\leq C_1 \left| \int_{\tilde{T}_{ijk}} v_h(x) \nabla_s \cdot \delta^{y, \Omega}(x) ds(x) \right|, \end{aligned}$$

where $\delta^{y, \Omega} = \|x^*\| \delta^{y^*, \Omega} \vec{n}_{y, \tilde{T}_{ijk}}$, with $y = \mathcal{P}(y^*)$.

Let us now introduce a variational formulation for the discrete Green's functions. Fix $y \in \mathbb{S}^2$, then we consider two kinds of regularized Green's functions $g_0^y, g_1^y \in C^\infty(\mathbb{S}^2)$ satisfying the following variational problems:

$$\mathcal{A}(v, g_0^y) = (v, \delta^y), \quad \text{for each } v \in H_0^1(\mathbb{S}^2), \quad (5.17a)$$

$$\mathcal{A}(v, g_1^y) = (v, \nabla_s \cdot \delta^{y, \Omega}), \quad \text{for each } v \in H_0^1(\mathbb{S}^2). \quad (5.17b)$$

Accordingly, we define $g_{0,h}^y, g_{1,h}^y \in \tilde{U}_h$ as the solutions for the finite element approximate problems

$$\mathcal{A}(w_h, g_{0,h}^y) = (w_h, \delta^y), \quad \text{for each } w_h \in \tilde{U}_h, \quad (5.18)$$

and

$$\mathcal{A}(w_h, g_{1,h}^y) = (w_h, \nabla_s \cdot \delta^{y, \Omega}), \quad \text{for each } w_h \in \tilde{U}_h. \quad (5.19)$$

The finite element approximation $g_{n,h}^y \in \tilde{U}_h$ with $n \in \{0, 1\}$ is taken to be the unique solution of problem

$$\mathcal{A}(w_h, g_n^y - g_{n,h}^y) = 0, \quad \text{for each } w_h \in \tilde{U}_h. \quad (5.20)$$

Notice that $g_{n,h}^y$ satisfies (5.15) from the structure of the space \tilde{U}_h for $n \in \{0, 1\}$. On the other hand, the term $g_n^y - g_{n,h}^y$ satisfies the error estimates in H^1 and L^2 -norm [13]. These discrete Green's functions appear because we will need some additional *a priori* estimates, which are well studied in the literature by [13, 38, 49, 52].

Lemma 5.8 *Let g_n^y be a discrete Green's function and $g_{n,h}^y \in \tilde{U}_h$ its finite element approximation with $n \in \{0, 1\}$. Then, we have:*

$$\|\nabla_s(g_1^y - g_{1,h}^y)\|_{L^1(\mathbb{S}^2)} \leq C, \quad (5.21a)$$

$$\|\nabla_s g_1^y\|_{L^1(\mathbb{S}^2)} + \|g_0^y\|_{W^{2,1}(\mathbb{S}^2)} \leq C|\ln h|, \quad (5.21b)$$

$$\|\nabla_s g_{0,h}^y\|_{L^2(\mathbb{S}^2)} \leq C|\ln h|^{1/2}. \quad (5.21c)$$

where C are positive generic constants independent of h . Here the factor $|\ln h|^{1/2}$ has order $\mathcal{O}(h^{-\eta})$ for $\eta \in (0, 1)$.

Proof We omit the details of (5.21a) and (5.21b), but we highlight that these proofs are detailed in [13, Lemma 3.3, pp. 819] and [38, Lemma 5.2, pp. 10]. About (5.21c), let $g_{0,h}^y \in \tilde{U}_h$ be the finite element approximation of g_0^y . From (5.18) follows that

$$\|\nabla_s g_{0,h}^y\|_{L^2(\mathbb{S}^2)}^2 = \mathcal{A}(g_{0,h}^y, g_{0,h}^y) = (g_{0,h}^y, \delta^y).$$

From discrete Sobolev inequality (see e.g., [7, Lemma 4.9.2, pp. 124] or [36, Lemma 3.12, pp. 527]), there exists a positive constant C , such that

$$\left|g_{0,h}^y(y)\right| \leq C|\ln h|^{1/2} \|\nabla_s g_{0,h}^y\|_{L^2(\mathbb{S}^2)}. \quad (5.22)$$

Therefore, we obtain (5.21c). \square

Now, we can show a weak stability condition of approximate solutions of (4.4) in the max-norm.

Proposition 5.9 *Assume that $u_h \in \tilde{U}_h$ is the unique solution of (4.12). Then, there exists a positive constant C , independent of h , such that*

$$\|u_h\|_{L^\infty(\mathbb{S}^2)} \leq C|\ln h|^{1/2} \|f\|_{L^2(\mathbb{S}^2)}.$$

Proof From Proposition 5.7, there exists δ^y supported in $\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h$ such that $|u_h(y)| = |(u_h, \delta^y)| = |\mathcal{A}(u_h, g_{0,h}^y)|$. In virtue of continuity of $\mathcal{A}(\cdot, \cdot)$ and Lemma 5.8, we have

$$|u_h(y)| = \left|\mathcal{A}\left(u_h, g_{0,h}^y\right)\right| \leq C\|\nabla_s u_h\|_{L^2(\mathbb{S}^2)} \|\nabla_s g_{0,h}^y\|_{L^2(\mathbb{S}^2)} \leq C|\ln h|^{1/2} \|\nabla_s u_h\|_{L^2(\mathbb{S}^2)}.$$

Finally, we get the desired inequality by applying Propositions 4.6 and 4.4. \square

We now state and show the main results of this section.

Theorem 5.10 Let $\tilde{\mathcal{V}}_h = \{x_i, \tilde{\mathcal{V}}_i\}_{i=1}^N$ be an almost uniform Voronoï-Delaunay decomposition on \mathbb{S}^2 . Assume that $f \in L^2(\mathbb{S}^2)$ satisfies (2.1) and the unique weak solution u of (2.2) belongs to $W^{2,\infty}(\mathbb{S}^2) \cap H^2(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$. Let $\bar{\mathcal{F}}_{ij}$ be the discrete flux defined in (4.2), such that the discrete problem (4.12) has a unique solution $u_h \in \tilde{\mathcal{U}}_h$. Then, there exists a positive constant C , independent of h , such that

$$\|\varepsilon_h\|_{L^\infty(\mathbb{S}^2)} \leq Ch |\ln h|^{1/2} (\|u\|_{H^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)}) + Ch^2 \|u\|_{W^{2,\infty}(\mathbb{S}^2)},$$

where $\varepsilon_h = u - u_h$.

Proof We shall fix $y \in \mathbb{S}^2$ and consider a discrete Green's function g_0^y satisfying (5.15) and the problem (5.17a). We also consider the finite element approximation $g_{0,h}^y \in \tilde{\mathcal{U}}_h$ satisfying the problem (5.20). By Proposition 5.7, there exists δ^y supported in $\tilde{\mathcal{T}}_{ijk} \in \tilde{\mathcal{T}}_h$, we then have

$$\begin{aligned} |(u - u_h)(y)| &\leq |(u - \tilde{\Pi}_h(u))(y)| + |(\tilde{\Pi}_h(u) - u_h)(y)| \\ &\leq |(u - \tilde{\Pi}_h(u))(y)| + \left| \int_{\mathbb{S}^2} (\tilde{\Pi}_h(u) - u_h)(x) \delta^y(x) ds(x) \right| \\ &\leq \|u - \tilde{\Pi}_h(u)\|_{L^\infty(\mathbb{S}^2)} + |\mathcal{A}(\tilde{\Pi}_h(u) - u_h, g_0^y)| \\ &\leq \|u - \tilde{\Pi}_h(u)\|_{L^\infty(\mathbb{S}^2)} + \left| \mathcal{A}(\tilde{\Pi}_h(u) - u_h, g_0^y - g_{0,h}^y) \right| \\ &\quad + \left| \mathcal{A}(\tilde{\Pi}_h(u) - u_h, g_{0,h}^y) \right| \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (5.23)$$

About I_1 , using Proposition 4.1 with $p = \infty$, we have

$$I_1 = \|u - \tilde{\Pi}_h(u)\|_{L^\infty(\mathbb{S}^2)} \leq Ch^2 \|u\|_{W^{2,\infty}(\mathbb{S}^2)}. \quad (5.24)$$

For I_2 , from (5.20), we obtain

$$I_2 = 0. \quad (5.25)$$

Finally, for I_3 , utilizing the continuity of $\mathcal{A}(\cdot, \cdot)$ and Proposition 4.1, Theorem 5.1 and inequality (5.21c) from Lemma 5.8, we have

$$\begin{aligned} I_3 &\leq C \|\nabla_s(\tilde{\Pi}_h(u) - u_h)\|_{L^2(\mathbb{S}^2)} \|\nabla_s g_{0,h}^y\|_{L^2(\mathbb{S}^2)} \\ &\leq C (\|\nabla_s(u - \tilde{\Pi}_h(u))\|_{L^2(\mathbb{S}^2)} + \|\nabla_s(u - u_h)\|_{L^2(\mathbb{S}^2)}) \|\nabla_s g_{0,h}^y\|_{L^2(\mathbb{S}^2)} \\ &\leq Ch |\ln h|^{1/2} (\|u\|_{H^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)}). \end{aligned} \quad (5.26)$$

Combining (5.24)–(5.26) and (5.23) for $h > 0$ small enough, we find

$$|(u - u_h)(y)| \leq Ch |\ln h|^{1/2} (\|u\|_{H^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)}) + Ch^2 \|u\|_{W^{2,\infty}(\mathbb{S}^2)}.$$

Finally, taking the maximum value leads to the desired result.

Notice that the finite volume scheme (4.4) is included in the estimate of H^1 -norm. Further, the result above is not optimal concerning the regularity required of the exact solution [22]. This excessive regularity can be removed as follows: put the restriction in the weak solution u of (2.2), belonging to $W^{2,\infty}(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$. Next, estimate a \max -norm for the tangential gradient of the solutions by using Propositions 4.1, 5.7, and Lemma 5.8. Finally, compute the error estimates of approximate solutions in the \max -norm.

In order to prove that, the following result provides a pointwise error estimate for the tangential gradient of the approximate solution.

Theorem 5.11 *Let $\tilde{\mathcal{V}}_h = \{x_i, \tilde{\mathcal{V}}_i\}_{i=1}^N$ be an almost uniform Voronoi-Delaunay decomposition on \mathbb{S}^2 . Assume that $f \in L^\infty(\mathbb{S}^2)$ satisfies (2.1) and that the unique weak solution u of (2.2) belongs to $W^{2,\infty}(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$. Let $\tilde{\mathcal{F}}_{ij}$ be a discrete flux (4.2), such that the discrete problem (4.12) has a unique solution $u_h \in \tilde{\mathcal{U}}_h$. Then, there exist positive constants C and h_0 independent of u , such that for $0 < h \leq h_0 < 1$,*

$$\|\nabla_s \varepsilon_h\|_{L^\infty(\mathbb{S}^2)} \leq Ch |\ln h| (\|u\|_{W^{2,\infty}(\mathbb{S}^2)} + \|f\|_{L^\infty(\mathbb{S}^2)}),$$

where $\varepsilon_h = u - u_h$.

Proof We will proceed via a duality argument: we fix $y \in \mathbb{S}^2$ and consider $\tilde{n}_{y,\tilde{\mathcal{T}}_{ijk}} \in \mathbb{T}_{y,\mathbb{S}^2}$ a tangent unit vector, and from Proposition 5.7, there exists $\delta^{y,\Omega}$ supported in $\tilde{\mathcal{T}}_{ijk}$. Let g_1^y be a discrete Green's function satisfying (5.15) and the variational problem (5.17b). Also, we consider the finite element approximation $g_{1,h}^y$ as the unique solution of (5.20). From the triangular inequality, we have

$$\begin{aligned} \left| \nabla_s(u - u_h)(y) \cdot \tilde{n}_{y,\tilde{\mathcal{T}}_{ijk}} \right| &\leq \left| \nabla_s(u - \tilde{\Pi}_h(u))(y) \cdot \tilde{n}_{y,\tilde{\mathcal{T}}_{ijk}} \right| \\ &\quad + \left| \nabla_s(\tilde{\Pi}_h(u) - u_h)(y) \cdot \tilde{n}_{y,\tilde{\mathcal{T}}_{ijk}} \right| \\ &\leq \|\nabla_s(u - \tilde{\Pi}_h(u))\|_{L^\infty(\mathbb{S}^2)} \\ &\quad + \left| \int_{\mathbb{S}^2} (\tilde{\Pi}_h(u) - u_h)(x) \nabla_s \cdot \delta^{y,\Omega}(x) ds(x) \right| \\ &\leq \|\nabla_s(u - \tilde{\Pi}_h(u))\|_{L^\infty(\mathbb{S}^2)} + |\mathcal{A}(\tilde{\Pi}_h(u) - u_h, g_1^y)| \\ &\leq \|\nabla_s(u - \tilde{\Pi}_h(u))\|_{L^\infty(\mathbb{S}^2)} + |\mathcal{A}(\tilde{\Pi}_h(u) - u_h, g_1^y - g_{1,h}^y)| \\ &\quad + |\mathcal{A}(\tilde{\Pi}_h(u) - u, g_{1,h}^y)| + |\mathcal{A}(u - u_h, g_{1,h}^y)| \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (5.27)$$

About I_1 , applying Proposition 4.1, we have

$$I_1 = \|\nabla_s(u - \tilde{\Pi}_h(u))\|_{L^\infty(\mathbb{S}^2)} \leq Ch \|u\|_{W^{2,\infty}(\mathbb{S}^2)}. \quad (5.28)$$

For I_2 , from (5.20), we obtain

$$I_2 = |\mathcal{A}(\tilde{\Pi}_h(u) - u_h, g_1^y - g_{1,h}^y)| = 0. \quad (5.29)$$

For I_3 , using the continuity of $\mathcal{A}(\cdot, \cdot)$, Proposition 4.1, the triangular inequality and equations (5.21b) and (5.21a) from Lemma 5.8, we have

$$\begin{aligned} I_3 &\leq C \|\nabla_s(u - \tilde{\Pi}_h(u))\|_{L^\infty(\mathbb{S}^2)} \|\nabla_s g_{1,h}^y\|_{L^1(\mathbb{S}^2)} \\ &\leq C \|\nabla_s(u - \tilde{\Pi}_h(u))\|_{L^\infty(\mathbb{S}^2)} \left(\|\nabla_s(g_1^y - g_{1,h}^y)\|_{L^1(\mathbb{S}^2)} + \|\nabla_s g_1^y\|_{L^1(\mathbb{S}^2)} \right) \\ &\leq Ch(1 + |\ln h|) \|u\|_{W^{2,\infty}(\mathbb{S}^2)}. \end{aligned} \quad (5.30)$$

Now about I_4 , by using the linearity of $\mathcal{A}(\cdot, \cdot)$ and adding up and subtracting the total fluxes $\overline{\mathcal{A}}_h(\cdot, \cdot)$ and $\tilde{\mathcal{A}}_h(\cdot, \cdot)$, we obtain

$$\begin{aligned} I_4 &\leq \left| \mathcal{A}(u, g_{1,h}^y) - \overline{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(g_{1,h}^y)) \right| + \left| \overline{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(g_{1,h}^y)) - \tilde{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(g_{1,h}^y)) \right| \\ &\quad + \left| \tilde{\mathcal{A}}_h(u_h, \tilde{\Gamma}_h(g_{1,h}^y)) - \mathcal{A}(u_h, g_{1,h}^y) \right| \\ &= I_{4,1} + I_{4,2} + I_{4,3}. \end{aligned} \quad (5.31)$$

For $I_{4,1}$, applying Lemmas 4.10 and 5.8 for $f \in L^\infty(\mathbb{S}^2)$ and $g_{1,h}^y \in \tilde{U}_h$, we find that

$$\begin{aligned} I_{4,1} &= \left| (f, g_{1,h}^y - \tilde{\Gamma}_h(g_{1,h}^y)) \right| \leq Ch \|f\|_{L^\infty(\mathbb{S}^2)} \|\nabla_s g_{1,h}^y\|_{L^1(\mathbb{S}^2)} \\ &\leq Ch(1 + |\ln h|) \|f\|_{L^\infty(\mathbb{S}^2)}. \end{aligned} \quad (5.32)$$

For $I_{4,2}$, by using Lemma 4.7 and collecting the inequalities (5.21a) and (5.21b) from Lemma 5.8, we arrive at

$$I_{4,2} \leq Ch^2 \|\nabla_s u_h\|_{L^\infty(\mathbb{S}^2)} \|\nabla_s g_{1,h}^y\|_{L^1(\mathbb{S}^2)} \leq Ch^2(1 + |\ln h|) \|\nabla_s u_h\|_{L^\infty(\mathbb{S}^2)}. \quad (5.33)$$

Similarly, for $I_{4,3}$, from Lemmas 4.9 and 5.8, we obtain

$$I_{4,3} \leq Ch^2 \|\nabla_s u_h\|_{L^\infty(\mathbb{S}^2)} \|\nabla_s g_{1,h}^y\|_{L^1(\mathbb{S}^2)} \leq Ch^2(1 + |\ln h|) \|\nabla_s u_h\|_{L^\infty(\mathbb{S}^2)}. \quad (5.34)$$

Notice that from triangle inequality, we obtain

$$\|\nabla_s u_h\|_{L^\infty(\mathbb{S}^2)} \leq \|\nabla_s(u - u_h)\|_{L^\infty(\mathbb{S}^2)} + \|\nabla_s u\|_{L^\infty(\mathbb{S}^2)}. \quad (5.35)$$

Thus, gathering all the estimates (5.32)–(5.35) into (5.31), follows that

$$I_4 \leq Ch(1 + |\ln h|) \|f\|_{L^\infty(\mathbb{S}^2)} + Ch^2(1 + |\ln h|) \left(\|\nabla_s(u - u_h)\|_{L^\infty(\mathbb{S}^2)} + \|\nabla_s u\|_{L^\infty(\mathbb{S}^2)} \right). \quad (5.36)$$

Finally, combining (5.36) with (5.27)–(5.30), for $h_0 \in \mathbb{R}_+$, such that $0 < h \leq h_0 < 1$ and by applying the maximum value, we find

$$\|\nabla_s(u - u_h)\|_{L^\infty(\mathbb{S}^2)} \leq Ch |\ln h| \left(\|u\|_{W^{2,\infty}(\mathbb{S}^2)} + \|f\|_{L^\infty(\mathbb{S}^2)} \right),$$

which leads to the desired result. \square

Now, we show error estimates for approximate solution in max-norm.

Theorem 5.12 *Under the assumptions of Theorem 5.11. Then, there exist positive constants C and h_0 independent of u , such that for $0 < h \leq h_0 < 1$,*

$$\|\varepsilon_h\|_{L^\infty(\mathbb{S}^2)} \leq Ch |\ln h| (\|u\|_{W^{2,\infty}(\mathbb{S}^2)} + \|f\|_{L^\infty(\mathbb{S}^2)}),$$

where $\varepsilon_h = u - u_h$.

Proof We proceed similarly to Theorem 5.11. We fix $y \in \mathbb{S}^2$, and from Proposition 5.7, there exists a smooth function δ^y supported in $\tilde{T}_{ijk} \in \tilde{\mathcal{T}}_h$. Let g_0^y be a discrete Green's function satisfying (5.15) and the variational problem (5.17a). We also consider the finite element approximation $g_{0,h}^y$ as a unique solution of the problem (5.20). By using triangular inequality, we have

$$\begin{aligned} |(u - u_h)(y)| &\leq |(u - \tilde{\Pi}_h(u))(y)| + |(\tilde{\Pi}_h(u) - u_h)(y)| \\ &\leq \|u - \tilde{\Pi}_h(u)\|_{L^\infty(\mathbb{S}^2)} + \left| \int_{\mathbb{S}^2} (\tilde{\Pi}_h(u) - u_h)(x) \delta^y(x) ds(x) \right| \\ &\leq \|u - \tilde{\Pi}_h(u)\|_{L^\infty(\mathbb{S}^2)} + |\mathcal{A}(\tilde{\Pi}_h(u) - u_h, g_0^y)| \\ &\leq \|u - \tilde{\Pi}_h(u)\|_{L^\infty(\mathbb{S}^2)} + |\mathcal{A}(\tilde{\Pi}_h(u) - u_h, g_0^y - g_{0,h}^y)| \\ &\quad + |\mathcal{A}(\tilde{\Pi}_h(u) - u, g_{0,h}^y)| + |\mathcal{A}(u - u_h, g_{0,h}^y)| \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (5.37)$$

For I_1 , from Proposition 4.1, we have

$$I_1 = \|u - \tilde{\Pi}_h(u)\|_{L^\infty(\mathbb{S}^2)} \leq Ch^2 \|u\|_{W^{2,\infty}(\mathbb{S}^2)}. \quad (5.38)$$

For I_2 , from (5.20), we have

$$I_2 = \mathcal{A}(\tilde{\Pi}_h(u) - u_h, g_0^y - g_{0,h}^y) = 0. \quad (5.39)$$

For I_3 , using the continuity of $\mathcal{A}(\cdot, \cdot)$, Proposition 4.1 and Lemma 5.8 with (5.21c), we obtain

$$I_3 = |\mathcal{A}(\tilde{\Pi}_h(u) - u, g_{0,h}^y)| \leq Ch |\ln h|^{1/2} \|u\|_{W^{2,\infty}(\mathbb{S}^2)}. \quad (5.40)$$

About I_4 , analogously to Theorem 5.11, by using the linearity of $\mathcal{A}(\cdot, \cdot)$ and the total fluxes $\tilde{\mathcal{A}}_h(\cdot, \cdot)$ and $\overline{\mathcal{A}}_h(\cdot, \cdot)$, we have

$$\begin{aligned} I_4 &\leq |\mathcal{A}(u, g_{0,h}^y) - \overline{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(g_{0,h}^y))| + |\overline{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(g_{0,h}^y)) - \tilde{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(g_{0,h}^y))| \\ &\quad + |\tilde{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(g_{0,h}^y)) - \mathcal{A}(u_h, g_{0,h}^y)| \\ &= I_{4,1} + I_{4,2} + I_{4,3}. \end{aligned} \quad (5.41)$$

For $I_{4,1}$, by using Lemmas 4.10 and 5.8, we find

$$I_{4,1} = \left| \mathcal{A}(u, g_{0,h}^y) - \bar{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(g_{0,h}^y)) \right| = \left| (f, g_{0,h}^y - \tilde{\mathcal{I}}_h(g_{0,h}^y)) \right| \leq Ch |\ln h| \|f\|_{L^\infty(\mathbb{S}^2)}. \quad (5.42)$$

For $I_{4,2}$, Lemma 4.7 yields

$$I_{4,2} = \left| \bar{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(g_{0,h}^y)) - \tilde{\mathcal{A}}_h(u_h, \tilde{\mathcal{I}}_h(g_{0,h}^y)) \right| \leq Ch^2 |\ln h| \|\nabla_s u_h\|_{L^\infty(\mathbb{S}^2)}.$$

Applying Theorem 5.11, there exists $h_0 \in \mathbb{R}_+$ such that for all $0 < h \leq h_0$, we have

$$\|\nabla_s u_h\|_{L^\infty(\mathbb{S}^2)} \leq Ch |\ln h| (\|u\|_{W^{2,\infty}(\mathbb{S}^2)} + \|f\|_{L^\infty(\mathbb{S}^2)}) + \|\nabla_s u\|_{L^\infty(\mathbb{S}^2)}.$$

Then

$$I_{4,2} \leq Ch^2 |\ln h| \|u\|_{W^{2,\infty}(\mathbb{S}^2)}. \quad (5.43)$$

As for $I_{4,3}$, using Lemma 4.7, we have

$$I_{4,3} \leq Ch^2 |\ln h| \|u\|_{W^{2,\infty}(\mathbb{S}^2)}. \quad (5.44)$$

Consequently, for $h > 0$ small enough and gathering (5.41) with (5.42)–(5.44), we have

$$I_4 \leq Ch |\ln h| \|f\|_{L^\infty(\mathbb{S}^2)}. \quad (5.45)$$

Finally, combining (5.37) with (5.38)–(5.40) and (5.45), we find

$$|(u - u_h)(y)| \leq Ch |\ln h| (\|u\|_{W^{2,\infty}(\mathbb{S}^2)} + \|f\|_{L^\infty(\mathbb{S}^2)}).$$

Therefore, applying the maximum value yields the expected result. \square

To end this section, we can get an additional estimate using a Voronoï-Delaunay decomposition SCVT.

Theorem 5.13 *Let $\tilde{\mathcal{V}}_h = \{x_i, \tilde{\mathcal{V}}_i\}_{i=1}^N$ be an almost uniform Spherical Centroidal Voronoï-Delaunay decomposition (SCVT) on \mathbb{S}^2 . Assume that $f \in H_0^1(\mathbb{S}^2)$ and that $u \in W^{2,\infty}(\mathbb{S}^2) \cap H^2(\mathbb{S}^2) \cap H_0^1(\mathbb{S}^2)$ is the unique solution of (2.2). Let $\bar{\mathcal{F}}_{ij}$ be the discrete flux defined in (4.2), such that the discrete problem (4.12) has a unique solution $u_h \in \tilde{\mathcal{U}}_h$. Then, there exist positive constants C and h_0 , independent of u , such that for $0 < h \leq h_0 < 1$,*

$$\|\varepsilon_h\|_{L^\infty(\mathbb{S}^2)} \leq Ch^2 \|u\|_{W^{2,\infty}(\mathbb{S}^2)} + Ch (\|u\|_{H^2(\mathbb{S}^2)} + \|f\|_{H^1(\mathbb{S}^2)}),$$

where $\varepsilon_h = u - u_h$.

Proof From triangular inequality, Proposition 4.1, Lemma 4.2 and Theorem 5.5, we then obtain

$$\begin{aligned}\|u - u_h\|_{L^\infty(\mathbb{S}^2)} &\leq \|u - \tilde{\Pi}_h(u)\|_{L^\infty(\mathbb{S}^2)} + \|\tilde{\Pi}_h(u) - u_h\|_{L^\infty(\mathbb{S}^2)} \\ &\leq Ch^2 \|u\|_{W^{2,\infty}(\mathbb{S}^2)} + Ch^{-1} \|\tilde{\Pi}_h(u) - u_h\|_{L^2(\mathbb{S}^2)} \\ &\leq Ch^2 \|u\|_{W^{2,\infty}(\mathbb{S}^2)} + Ch (\|u\|_{H^2(\mathbb{S}^2)} + \|f\|_{H^1(\mathbb{S}^2)}).\end{aligned}$$

For h small enough, the proof is complete. \square

6 Numerical example and final remarks

This section illustrates an example of the FV approach of the Laplace-Beltrami operator using the recursive Voronoï-Delaunay decomposition. We consider three types of grids: the non-optimized grid (NOPT) and two of the most used grid optimizations in the literature [46], the Heikes and Randall optimized grids (HR95), proposed in [30, 31], and the *Spherical Centroidal Voronoï Tessellations* (SCVT) described by Du and collaborators in [18, 19]. We consider the SCVT grids with constant density function ($\rho = 1$). To verify the error estimate ε_h , we shall use the example defined in [31]. The exact solution u in geographic coordinates (ϕ, θ) is defined as

$$u(\phi, \theta) = \cos \theta \cos^4 \phi, \quad (6.1)$$

and the forcing source as,

$$\begin{aligned}f(\phi, \theta) &= -\cos \theta \cos^2 \theta [\cos^2 \phi - 4 \sin \phi \cos \phi \sin \phi \cos \phi \\ &\quad - 12 \cos^2 \phi + 16 \cos^2 \phi \cos^2 \phi] / \cos^2 \phi,\end{aligned} \quad (6.2)$$

where $\phi \in [-\pi/2, \pi/2]$ is the latitude and $\theta \in [-\pi, \pi]$ is the longitude.

Approximate solutions were obtained using the finite volume scheme (4.4). Figure 5 shows error estimates and convergence rates of the approximate solution of the problem (6.2) in H^1 , L^2 , $W^{1,\infty}$ and max-norm using three types of grids and different refinement levels. The numerical convergence rate CR with respect to the norm $\|\cdot\|_\star$ is given as

$$CR = \frac{|\ln \|\varepsilon_n\|_\star - \ln \|\varepsilon_{n-1}\|_\star|}{\ln 2}, \quad \text{for } n = 2, \dots, N,$$

where $\varepsilon_n = u - u_h$ denotes the error of the n -th level.

Firstly, by using grid NOPT, we observe that the numerical convergence rate is just about $\mathcal{O}(h)$ in H^1 -norm and matches with the theoretical convergence rate predicted in Theorem 5.1. Analogously, from Theorem 5.11, we have that the theoretical prediction for the solution error is $\mathcal{O}(h |\ln h|)$ in $W^{1,\infty}$ -norm. At the same time, the obtained logarithmic factor is not detected numerically.

Furthermore, the numerical convergence rates for problem (6.2) indicate that the error given in L^2 and max-norm tends to be quadratic order as we refine the grid

NOPT. In general, the difference between the right-hand sides of the variational problems (2.4) and (4.12), when using non-optimized Voronoï-Delaunay decomposition, is a dominant factor for optimal convergence rate (just about $\mathcal{O}(h)$), as predicted in Theorems 5.1 and 5.12 respectively. To our knowledge, no existing analytical results confirm the quadratic order in these norms for general Voronoï-Delaunay tessellations.

However, observe that the numerical convergence rate for SCVT is just about $\mathcal{O}(h^2)$ in the L^2 -norm, as had been shown by Du in [20]. Here, we modify the proof by using a minimal regularity requirement in the exact solution and highlight that the quadratic order error estimates depend on the geometric criterion of the SCVT. The numerical convergence rates in max-norm is also $\mathcal{O}(h^2)$. This latter case is under study, and results will be presented elsewhere. Note also that the best behavior of error estimates is given

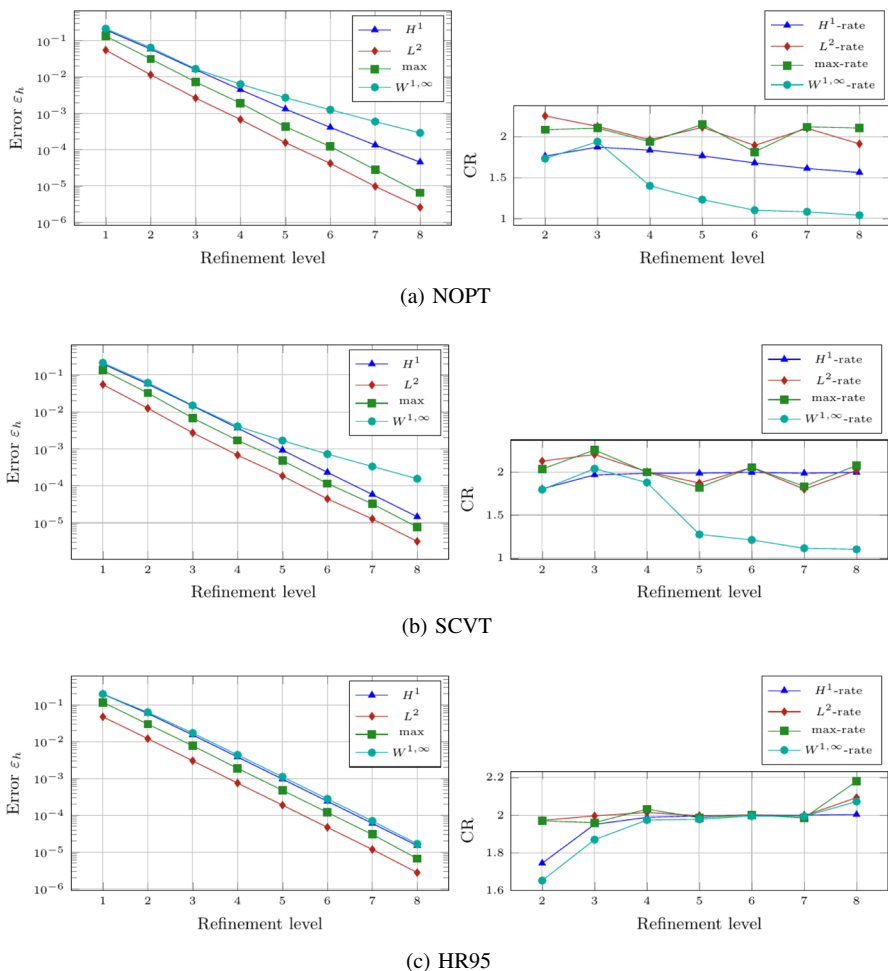


Fig. 5 Errors ε_h and numerical convergence rates CR in the finite volume approximation for problem (6.2) with exact solution (6.1) in H^1 , L^2 , max and $W^{1,\infty}$ -norm using grids NOPT, SCVT and HR95 with different refinement levels

in H^1 -norm in both optimized grids. Thus, there exists a degree of superconvergence of the approach on gradients. To date, there seem to be no existing theoretical criteria that prove these behaviors, and consequently, this brings a good challenge for future research. Extensions of the analysis to HR95 grids are our current investigation.

Acknowledgements This work is in memory of Saulo R. M. Barros, who participated in this work but tragically passed away in July 2021, prior to its conclusion. The work presented here was supported by FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo) through grant 2016/18445-7 and 2021/06176-0 and also by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES), Finance Code 001. Finally, we thank the anonymous referees for carefully reading the manuscript and for perceptive comments, which have led to an improvement in the quality of the paper.

Declarations

Conflict of interest Each author has contributed substantially to conducting the underlying research and writing this manuscript. Furthermore, they have no financial or other conflicts of interest to disclose.

References

1. Aubin, T.: Some Nonlinear Problems in Riemannian Geometry. Springer Science & Business Media (2013)
2. Augenbaum, J.M., Peskin, C.S.: On the construction of the Voronoi mesh on a sphere. *J. Comput. Phys.* **59**(2), 177–192 (1985)
3. Barbeiro, S.: Supraconvergent cell-centered scheme for two dimensional elliptic problems. *Appl. Numer. Math. IMACS J.* **59**(1), 56–72 (2009)
4. Baumgardner, J.R., Frederickson, P.O.: Icosahedral discretization of the two-sphere. *SIAM J. Numer. Anal.* **22**(6), 1107–1115 (1985)
5. Bessemoulin-Chatard, M., Chainais-Hillairet, C., Filbet, F.: On discrete functional inequalities for some finite volume schemes. *IMA J. Numer. Anal.* **35**(3), 1125–1149 (2015)
6. Bouche, D., Ghidaglia, J.-M., Pascal, F.: Error estimate and the geometric corrector for the upwind finite volume method applied to the linear advection equation. *SIAM J. Numer. Anal.* **43**(2), 578–603 (2005)
7. Brenner S., Scott R.: The Mathematical Theory of Finite Element Methods, vol. 15. Springer Science & Business Media (2007)
8. Chen, Z., Li, R., Zhou, A.: A note on the optimal L^2 -estimate of the finite volume element method. *Adv. Comput. Math.* **16**(4), 291–303 (2002)
9. Chou, S.-H., Kwak, D.Y., Li, Q.: L^p error estimates and superconvergence for covolume or finite volume element methods. *Numer. Methods Partial Differential Equations* **19**(4), 463–486 (2003)
10. Chou, S.-H., Li, Q.: Error estimates in L^2 , H^1 and L^∞ in covolume methods for elliptic and parabolic problems: a unified approach. *Math. Comput.* **69**(229), 103–120 (2000)
11. Chou, S.-H., Ye, X.: Superconvergence of finite volume methods for the second order elliptic problem. *Comput. Methods Appl. Mech. Eng.* **196**(37), 3706–3712 (2007)
12. Ciarlet, P.G.: The finite element method for elliptic problems. *SIAM* (2002)
13. Demlow, A.: Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces. *SIAM J. Numer. Anal.* **47**(2), 805–827 (2009)
14. Despres, B.: Lax theorem and finite volume schemes. *Math. Comput.* **73**(247), 1203–1234 (2004)
15. Diskin, B., Thomas, J.L.: Notes on accuracy of finite-volume discretization schemes on irregular grids. *Appl. Numer. Math. IMACS J.* **60**(3), 224–226 (2010)
16. Droniou, J.: Introduction to discrete functional analysis techniques for the numerical study of diffusion equations with irregular data. *ANZIAM J. Electron. Suppl.* **56**(C), c101–c127 (2014)
17. Du, Q., Faber, V., Gunzburger, M.: Centroidal Voronoi tessellations: applications and algorithms. *SIAM Rev.* **41**(4), 637–676 (1999)

18. Du, Q., Gunzburger, M.D., Ju, L.: Constrained centroidal Voronoi tessellations for surfaces. *SIAM J. Sci. Comput.* **24**(5), 1488–1506 (2003)
19. Du, Q., Gunzburger, M.D., Ju, L.: Voronoi-based finite volume methods, optimal Voronoi meshes, and PDEs on the sphere. *Comput. Methods Appl. Mech. Eng.* **192**(35), 3933–3957 (2003)
20. Du, Q., Ju, L.: Finite volume methods on spheres and spherical centroidal Voronoi meshes. *SIAM J. Numer. Anal.* **43**(4), 1673–1692 (2005)
21. Dziuk, G., Elliott, C.M.: Finite element methods for surface PDEs. *Acta Numer.* **22**, 289–396 (2013)
22. Ewing, R.E., Lin, T., Lin, Y.: On the accuracy of the finite volume element method based on piecewise linear polynomials. *SIAM J. Numer. Anal.* **39**(6), 1865–1888 (2002)
23. Eymard, R., Gallouët, T., Herbin, R.: A cell-centered finite-volume approximation for anisotropic diffusion operators on unstructured meshes in any space dimension. *IMA J. Numer. Anal.* **26**(2), 326–353 (2006)
24. Eymard, R., Gallouët, T., Herbin, R.: Finite volume approximation of elliptic problems and convergence of an approximate gradient. *Appl. Numer. Math. IMACS J.* **37**(1–2), 31–53 (2001)
25. Gallouët, T., Herbin, R., Vignal, M.H.: Error estimates on the approximate finite volume solution of convection diffusion equations with general boundary conditions. *SIAM J. Numer. Anal.* **37**(6), 1935–1972 (2000)
26. Gärtner, K., Kamenski, L.: Why do we need Voronoi cells and Delaunay meshes? essential properties of the Voronoi finite volume method. *Comput. Math. Math. Phys.* **59**(12), 1930–1944 (2019)
27. Giraldo, F.X.: Lagrange-Galerkin methods on spherical geodesic grids. *J. Comput. Phys.* **136**(1), 197–213 (1997)
28. Glitzky, A., Griepentrog, J.A.: Discrete sobolev-poincaré inequalities for Voronoi finite volume approximations. *SIAM J. Numer. Anal.* **48**(1), 372–391 (2010)
29. Hebey, E.: Sobolev Spaces on Riemannian Manifolds, vol. 1635. Springer Science & Business Media (1996)
30. Heikes, R.: Numerical integration of the shallow-water equations on a twisted icosahedral grid. Part I: Basic design and results of tests. *Mon. Weather Rev.* **123**(6), 1862–1880 (1995)
31. Heikes, R., Randall, D.A.: Numerical integration of the shallow-water equations on a twisted icosahedral grid. Part II. A detailed description of the grid and an analysis of numerical accuracy. *Mon. Weather Rev.* **123**(6), 1881–1887 (1995)
32. Hjelle, Ø., Dæhlen, M.: Triangulations and Applications. Springer Science & Business Media (2006)
33. Jianguo, H., Shitong, X.: On the finite volume element method for general self-adjoint elliptic problems. *SIAM J. Numer. Anal.* **35**(5), 1762–1774 (1998)
34. Ju, L., Du, Q.: A finite volume method on general surfaces and its error estimates. *J. Math. Anal. Appl.* **352**(2), 645–668 (2009)
35. Ju, L., Tian, L., Wang, D.: A posteriori error estimates for finite volume approximations of elliptic equations on general surfaces. *Comput. Methods Appl. Mech. Eng.* **198**(5–8), 716–726 (2009)
36. Kovács, B., Power Guerra, C.A.: Maximum norm stability and error estimates for the evolving surface finite element method. *Numer. Methods Partial Differential Equations* **34**(2), 518–554 (2018)
37. Kreiss, H.-O., Manteuffel, T.A., Swartz, B., Wendroff, B., White, A.B.: Supra-convergent schemes on irregular grids. *Math. Comput.* **47**(176), 537–554 (1986)
38. Kröner, H.: Approximative Green's functions on surfaces and pointwise error estimates for the finite element method. *Comput. Methods Appl. Math.* **17**(1), 51–64 (2017)
39. Leykekhman, D., Li, B.: Maximum-norm stability of the finite element Ritz projection under mixed boundary conditions. *Calcolo* **54**(2), 541–565 (2017)
40. Leykekhman, D., Li, B.: Weak discrete maximum principle of finite element methods in convex polyhedra. *Math. Comp.* **90**(327), 1–18 (2021)
41. Li, B.: Maximum-norm stability of the finite element method for the Neumann problem in nonconvex polygons with locally refined mesh. *Math. Comp.* **91**(336), 1533–1585 (2022)
42. Li, R., Chen, Z., Wu, W.: Generalized Difference Methods for Differential Equations: Numerical Analysis of Finite Volume Methods. CRC Press (2000)
43. Lin, Y., Liu, J., Yang, M.: Finite volume element methods: an overview on recent developments. *Int. J. Numer. Anal. Model. Ser. B* **4**(1), 14–34 (2013)
44. Manteuffel, T.A., White, A.B.: The numerical solution of second-order boundary value problems on nonuniform meshes. *Math. Comput.* **47**(176), 511–535 (1986)
45. Mishev, I.D.: Finite volume methods on Voronoi meshes. *Numer. Methods for Partial Differential Equations* **14**(2), 193–212 (1998)

46. Miura, H., Kimoto, M.: A comparison of grid quality of optimized spherical hexagonal-pentagonal geodesic grids. *Mon. Weather Rev.* **133**(10), 2817–2833 (2005)
47. Omnes, P.: On the second-order convergence of a function reconstructed from finite volume approximations of the Laplace equation on Delaunay-Voronoi meshes. *ESAIM: Math. Model. Numer. Anal.* **45**(4), 627–650 (2011)
48. Pascal, F.: On supra-convergence of the finite volume method for the linear advection problem. In: Paris-Sud Working Group on Modelling and Scientific Computing 2006–2007, vol. 18 of *ESAIM Proc.*, pp. 38–47. EDP Sci., Les Ulis (2007)
49. Rannacher, R., Scott, R.: Some optimal error estimates for piecewise linear finite element approximations. *Math. Comput.* **38**(158), 437–445 (1982)
50. Renka, R.J.: Algorithm 773: SSRFPACK: interpolation of scattered data on the surface of a sphere with a surface under tension. *ACM Trans. Math. Softw. (TOMS)* **23**(3), 435–442 (1997)
51. Sadourny, R., Arakawa, A., Mintz, Y.: Integration of the nondivergent barotropic vorticity equation with an icosahedral-hexagonal grid for the sphere. *Mon. Weather Rev.* **96**(6), 351–356 (1968)
52. Schatz, A.H., Wahlbin, L.B.: Interior maximum-norm estimates for finite element methods. II. *Math. Comput.* **64**(211):907–928 (1995). MR1297478
53. Schatz, A.: Pointwise error estimates and asymptotic error expansion inequalities for the finite element method on irregular grids: Part I. Global estimates. *Math. Comput. Am. Math. Soc.* **67**(223), 877–899 (1998)
54. Scott, R.: Optimal L^∞ estimates for the finite element method on irregular meshes. *Math. Comput.* **30**(136):681–697 (1976). MR0436617
55. Stuhne, G.R., Peltier, W.R.: New icosahedral grid-point discretizations of the shallow water equations on the sphere. *J. Comput. Phys.* **148**(1), 23–58 (1999)
56. Williamson, D.L.: Integration of the barotropic vorticity equation on a spherical geodesic grid. *Tellus* **20**(4), 642–653 (1968)
57. Wu, H., Li, R.: Error estimates for finite volume element methods for general second-order elliptic problems. *Numer. Methods for Partial Differential Equations* **19**(6), 693–708 (2003)
58. Zhang, T.: Superconvergence of finite volume element method for elliptic problems. *Adv. Comput. Math.* **40**(2), 399–413 (2014)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.